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ON POLYNOMIAL EXPANSION OF MULTIVALENT FUNCTIONS

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ABSTRACT. Coefficient bounds for mean p-valent functions, whose expansion in an ellipse has a Jacobi polynomial series, are given in this paper.

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1. INTRODUCTION.

Let $E_o = \{z = \cosh(s_o+i\tau),\ 0 \le \tau < 2\pi,\ s_o = \tanh^{-1}(b/a),\ a > b > 0\}$ be a fixed ellipse whose foci are ±1. Let also r_o = a+b be the sum of the semi-axis of E_o. It is known (Szegö [1], Theorem 9.1.1], see also p. 245) that a function f(z) which is regular in $Int(E_o)$ (this means the interior of E_o) has an expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(z) \tag{1.1}$$

where here and throughout this paper $\alpha,\beta > -1$. This expansion converges locally uniformly in $Int(E_o)$. In [2] the author has given some coefficient bounds for functions mean p-valent and has an expansion in terms of Chebyshev polynomials in $Int(E_o)$. Such polynomials are generated by the special case $\alpha = \beta = -1/2$ in Jacobi polynomials. Other special cases of interest are the Legendre and the altraspherical polynomials generated by $\alpha = \beta = 0$ and $\alpha = \beta$ respectively [1, p. 80-89].

In this paper we generalize results given in [2] to functions of the form (1.1) and mean p-valent in $Int(E_o)$. In view of [2] we call f(z) mean p-valent in $Int(E_o)$ if

$$W(R,f) = (1/\pi) \int_0^R \int_0^{2\pi} n(\rho e^{i\phi}, f, Int(E_o))\rho d\rho d\phi \le pR^2$$

where $0 < R < \infty$ and $n(\rho e^{i\phi}, f, Int(E_o))$ denotes the number of roots of the equation f(z) w in Interior E_o, multiplicity being take into account.

We first recall from [2]:

THEOREM A. Let f(z) be mean p-valent in $Int(E_o)$. Then for $z = \cosh(s+i\tau)$, exp(s) = r and $1 < r < r_o$ we have

$$|f(z)| = O(1)\ (1-r/r_o)^{-2p}$$

where O(1) depends on a,b and f only.

THEOREM B. Let $f(z)$ be mean p-valent in $\mathrm{Int}(E_o)$ and $M(r,f) \leq C(1-r/r_o)^{-\gamma}$ where $c,\gamma > 0$ and $M(r,f) = \max\{|f(z)| : z \in \mathrm{Int}(E_o)\}$. Set $z = \cosh(s+i\tau)$, $\exp(s)=r$, $1 < r < r_o$ and

$$I_1(r,f') = (1/2\pi)\int_0^{2\pi} |f'(\cosh(s+i\tau))||\sinh(s+i\tau)|dt.$$

Then as $r \to r_o$ we have

$$I_1(r,f') = \begin{cases} O(1)\ (1-r/r_o)^{-\gamma}, & (\gamma > 1/2), \\ O(1)\ (1-r/r_o)^{-1/2}\log(1/(1-r/r_o)), & (\gamma = 1/2), \\ o(1)\ (1-r/r_o)^{-1/2}, & (\gamma < 1/2), \end{cases}$$

where $O(1)$ and $o(1)$ depend on a,b,γ and f only.

PROOF OF THEOREM B. Using Schwarz's inequality we have

$$I_1(r,f') \leq [(1/2\pi)\int_0^{2\pi} |f'(\cosh(s+i\tau)|^2|f(\cosh(s+i\tau))|^{\lambda-2}|\sinh(s+i\tau)|^2d\tau)^{1/2}]^{1/2}$$
$$\times [(1/2\pi)\int_0^{2\pi} |f(\cosh(s+i\tau))|^{2-\lambda}d\tau)^{1/2}]^{1/2}$$

where $0 \leq \lambda < 2$. Theorem B now follows in the same way as estimating inequality (14) of [2] by using [2, Lemmas 3 and 4].

We now need a suitable coefficient formula.

LEMMA 1.1. Let $f(z) = \sum_{n=0}^{\infty} a_n p_n^{(\alpha,\beta)}(z)$ be regular in $\mathrm{Int}(E_o)$ and

$E = \{z = \cosh(s+i\tau),\ 0 \leq \tau < 2\pi\}$. Then for a fixed s so that $0 < s < s_o$ we have

$$a_n = (K_n^{(\alpha,\beta)}/h_n^{(\alpha,\beta)})(1/2\pi i)\int_E \frac{f(z)}{z^{n+1}}dz, \qquad (n \geq 0), \tag{1.2}$$

$$\tfrac{1}{2}(n+\alpha+\beta+1)a_n = (K_n^{(\alpha+1,\beta+1)}/h_{n-1}^{(\alpha+1,\beta+1)})(1/2\pi i)\int_E \frac{f'(z)}{z^n}dz, \quad (n \geq 1) \tag{1.3}$$

where $K_n^{(\alpha,\beta)} = 2^{n+\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)/\Gamma(2n+\alpha+\beta+2)$ and

$h_n^{(\alpha,\beta)} = 2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)/(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)$.

We note here, using Stirling's formula from Titmarsh [3, p. 57], that

$$K_n^{(\alpha,\beta)}/h_n^{(\alpha,\beta)} = O(1)n^{1/2}/2^n \tag{1.4}$$

as $n \to \infty$, where $O(1)$ depends on α,β only.

PROOF OF LEMMA. We have from [1, p. 245] that

$$a_n = \{\pi i h_n^{(\alpha,\beta)}\}^{-1}\int_E (z-1)^{\alpha}(z+1)^{\beta}Q_n^{(\alpha,\beta)}(z)f(z)dz \tag{1.5}$$

where $n = 0,1,2,\ldots$.

We now see from [1, Theorem 4.61.2], (see also Erdelyi, Magnus, Oberhettinger and Tricomi [4, p. 171], and Freud [5, p.44] that

$$(z-1)^{\alpha}(z+1)^{\beta}Q_n^{(\alpha,\beta)}(z) = (1/2)\sum_{k=0}^{\infty}\frac{1}{z^{k+1}}\int_{-1}^{1}(1-t)^{\alpha}(1+t)^{\beta}t^kp_n^{(\alpha,\beta)}(t)dt$$
$$= K_n^{(\alpha,\beta)}/2z^{n+1}, \tag{1.6}$$

where $K_n^{(\alpha,\beta)}$ is as defined above. In connection with this, see the argument used in the proof of formula (4.3.3) of [1, p.67].

Using (1.6) in (1.5) we immediately deduce (1.2).

Now differentiating (1.1) we see from equation (4.21.7) of [1] that

$$f'(z) = \sum_{n=1}^{\infty} \tfrac{1}{2}(n+\alpha+\beta+1)a_n P_{n-1}^{(\alpha+1,\beta+1)}(z).$$

Again, as in the proof of (1.2), we deduce from this and [1, p. 245] for $n \geq 1$, that

$$\tfrac{1}{2}(n+\alpha+\beta+1)a_n = \{\pi i\, h_{n-1}^{(\alpha+1,\beta+1)}\}^{-1} \int_E (z-1)^{\alpha+1}(z+1)^{\beta+1}Q_{n-1}^{(\alpha+1,\beta+1)}(z)f'(z)dz$$
$$= (K_{n-1}^{(\alpha+1,\beta+1)}/h_{n-1}^{(\alpha+1,\beta+1)})(1/2\pi i) \int_E \frac{f'(z)}{z^n}\,dz$$

where we have used the equation $(z-1)^{\alpha+1}(z+1)^{\beta+1}Q_{n-1}^{(\alpha+1,\beta+1)}(z) = K_{n-1}^{(\alpha+1,\beta+1)}/2z^n$ which is deduced as in (1.6). This is equation (1.3) and the proof of the lemma is now complete.

2. MAIN THEOREM.

THEOREM 2.1. Let $f(z) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(z)$ be mean p-valent in $\mathrm{Int}(E_o)$ and $M(r,f) < C(1-r/r_o)^{-\gamma}$ where C, $\gamma > 0$ and $M(r,f)$ is as defined above. Then, as $n \to \infty$ we have

$$|a_n| = r_o^{-n} \begin{cases} O(1)n^{\gamma-1/2}, & (\gamma < 1/2), \\ O(1)\ (\log n), & (\gamma = 1/2), \\ o(1), & (\gamma < 1/2), \end{cases}$$

where O(1) and o(1) depend on $a, b, \alpha, \beta, \gamma$ and f only.

PROOF OF THEOREM 2.1. From (1.3) and Theorem B we deduce, using the bounds $|\sinh(s+i\tau)| \geq \sinh s$, $|\cosh(s+i\tau)| \leq \cosh s$ and (1.4), that

$$\tfrac{1}{2}(n+\alpha+\beta+1)|a_n| \leq (K_{n-1}^{(\alpha+1,\beta+1)}/h_{n-1}^{(\alpha+1,\beta+1)})(\cosh I_1(r,f')/\sinh^n s)$$
$$\leq (K_{n-1}^{(\alpha+1,\beta+1)}/h_{n-1}^{(\alpha+1,\beta+1)})(2^n I_1(r,f')/r^n(1-1/r))$$

$$|a_n| = r_o^{-n} \begin{cases} O(1)n^{\gamma-1/2}, & (\gamma > 1/2), \\ O(1)(\log n), & (\gamma = 1/2), \\ o(1), & (\gamma < 1/2), \end{cases}$$

where we have chosen $r = ((n-1)/n)r_o$ and provided that $1-n/(n-1)r_o > 0$. This completes the proof of Theorem 2.1.

COROLLARY 2.1. Let $f(z) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(z)$ be mean p-valent in $\mathrm{Int}(E_o)$. Then, as $n \to \infty$ we have

$$|a_n| = r_o^{-n} \begin{cases} O(1)n^{2p-1/2}, & (p > 1/4), \\ O(1)\ (\log n), & (p = 1/4), \\ o(1), & (p < 1/4), \end{cases}$$

where O(1) and o(1) depend on a,b,α,β,p and f only. In view of Theorem A, the proof of Corollary 2.1 follows by setting $\gamma = 2p$ in Theorem 2.1.

COROLLARY 2.2. Let $f(z) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(z)$ be univalent in $\mathrm{Int}(E_o)$. Then as $n \to \infty$ we have

$$|a_n| = O(1)n^{3/2}r_o^{-n}$$

where O(1) depends on α,b,α,β and f only.

This corollary follows upon setting p = 1 in Corollary 2.1.

REMARK. Using the formula (4.21.2) of [1] and the argument used in [2, Remark 2] we see by setting $z = \xi \cosh s_o$ where $|\xi| = |\cos\tau + i\tanh s_o \sin\tau| < 1$ that

$$f(\xi\cosh s_o) = \sum_{n=0}^{\infty} \frac{\Gamma(2n+\alpha+\beta+1)}{n!\,\Gamma(n+\alpha+\beta+1)} a_n \left(\frac{\cosh s_o}{2}\right)^n \{(\xi - 1/\cosh s_o)^n$$

$$+ c_1(\xi - 1/\cosh s_o)^{n-1} + \dots + c_n/\cosh^n s_o\}$$

$$= \sum_{n=0}^{\infty} \hat{a}_n \hat{P}_n^{(\alpha,\beta)}(\xi)$$

where

$$\hat{P}_n^{(\alpha,\beta)}(\xi) = (\xi - 1/\cosh s_o)^n + c_1(\xi - 1/\cosh s_o)^{n-1} + \dots + c_n/\cosh^n s_o$$

and

$$\hat{a}_n = \Gamma(2n+\alpha+\beta+1)a_n \cosh^n s_o / 2^n \Gamma(n+1)\Gamma(n+\alpha+\beta+1).$$

Using this and Stirling's formula and letting $r_o \to \infty$ we see that Theorem 2.1 and Corollaries 2.1 and 2.2 correspond to analogous results for the unit disk (see Hayman [6]).

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