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ON RADII OF CONVEXITY AND STARLIKENESS OF SOME CLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT. Let $P[A,B]$, $-1 \le B < A \le 1$, be the class of functions p such that $p(z)$ is subordinate to $\frac{1+Az}{1+Bz}$. Let $P(\alpha_1)$ be the class of functions with positive real part greater than $\alpha_1, 0 \le \alpha_1 \le 1$. It is clear that $P[A,B] \subset P\left(\frac{1-A}{1-B}\right) \subset P[1,-1]$. The principal results in this paper are the determination of the radius of β-starlikeness and β-convexity of $f(z)$ with $\beta = \frac{1-A}{1-B}$, when $f(z)$ is restricted to certain classes of univalent and analytic functions related with $P[A,B]$.

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1. INTRODUCTION.

Let f be analytic in $E = \{z : |z| < 1\}$, and be given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n . \tag{1.1}$$

A function g, analytic in E, is called subordinate to a function G if there exists a Schwarz function $w(z)$, $w(z)$ analytic in E with $w(0) = 0$ and $|w(z)| < 1$ in E, such that $g(z) = G(w(z))$.

In [1], Janowski introduced the class $P[A,b]$. For A and B, $-1 \le B < A \le 1$, a function p, analytic in E with $p(0) = 1$ belongs to the class $P[A,B]$ if $p(z)$ is subordinate to $\frac{1+Az}{1+Bz}$.

Also $C[A,B]$ and $S^*[A,B]$ denote the classes of functions, analytic in E and given by (1.1) such that $\frac{(zf'(z))'}{f(z)} \varepsilon P[A,B]$ and $\frac{zf'(z)}{f(z)} \varepsilon P[A,B]$ respectively. For $A = 1$, and $B = -1$, we note that $C[1,-1] \equiv C$ and $S^*[1,-1] \equiv S^*$, the classes of convex and starlike functions in ε. Also $S^*[A,B] \subset S^*\left(\frac{1-A}{1-B}\right) \subset S^*[1,-1]$ and $C[A,B] \subset C\left(\frac{1-A}{1-B}\right) \subset C[1,-1]$, where $S^*\left(\frac{1-A}{1-B}\right)$ and $C\left(\frac{1-A}{1-B}\right)$ denote the classes of starlike and convex functions of order $\frac{1-A}{1-B}$ respectively. These classes were first introduced by Robertson in [2].

A function f, analytic in E and given by (1.1), is said to be in the class $R_k[A,B]$, $-1 \le B < A \le 1$, if and only if

Hence

$$\frac{(zf'(z))'}{f'(z)}-\frac{1-A}{1-B}=p(z)+\frac{zp'(z)}{p(z)}-\frac{1-A}{1-B}$$

Using Lemma 2.3 for $\alpha = 1 = \beta$, we have for $R_1 \le R_2$

$$Re\left[\frac{(zf'(z))'}{f'(z)}-\frac{1-A}{1-B}\right]\ge\frac{1-(3A-B)r+A^2r^2}{(1-Ar)(1-Br)}-\frac{1-A}{1-B}$$

$$=\frac{A-B}{1-B}\left[\frac{1-(2+A-B)r+Ar^2}{(1-Ar)(1-Br)}\right],$$

and this implies that $Re\left[\frac{(zf'(z))'}{f'(z)}-\frac{1-A}{1-B}\right]\ge 0$ for $|z|<r_0$, where r_0 is given by (3.1). The inequality $R_1 < R_2$ is satisfied whenever $T(r)=1-(2+A-B)r+Ar^2\ge 0$. But $T(0)=1>0$ and $T(1)=B-1<0$. So $T(r)$ has at least one root in (0,1). Let r_0, given by (3.1) be that root of $T(r)=0$. Then in $[0,r_0), R_1 < R_2$ and hence $f\,\varepsilon\, C\left(\frac{1-A}{1-B}\right)$ for all z with $|z| = r \le r_0 < 1$.

This result is sharp for the function $f_0\,\varepsilon\, S^*[A,B]$ such that

$$\frac{zf_0'(z)}{f_0(z)}=\frac{1+Az}{1+Bz}$$

THEOREM 3.2. Let $g\,\varepsilon\, S^*[A,B]$ and let $\frac{zf'(z)}{g(z)}\,\varepsilon\, P[A,B]$. Then $f\,\varepsilon\, C\left(\frac{1-A}{1-B}\right)$ for $|z|<r_0$, where r_0 is given by (3.1).

PROOF. $zf'(z)=g(z)p(z)$, $\qquad p\,\varepsilon\, P[A,B]$.

This gives us

$$\frac{(zf'(z))'}{f'(z)}=\frac{zg'(z)}{g(z)}+\frac{zp'(z)}{p(z)}$$

Applying the usual inequalities, we obtain

$$Re\left[\frac{(zf'(z))'}{f'(z)}-\frac{1-A}{1-B}\right]\ge\frac{1-Ar}{1-Br}-\frac{(A-B)r}{(1-Ar)(1-Br)}-\frac{1-A}{1-B}$$

$$=\frac{(A-B)[1-(2+A-B)r+Ar]}{(1-B)(1-Ar)(1-Br)}$$

Hence we obtain the required result that $f\,\varepsilon\, C\left(\frac{1-A}{1-B}\right)$ for $|z|<r_0$ and r_0 is given by (3.1).

THEOREM 3.3. Let $g\,\varepsilon\, S^*[A,B]$ and $\frac{zf'(z)}{g(z)}\,\varepsilon\, P[A,B]$. Then $\frac{(zf'(z))'}{g'(z)}\,\varepsilon\, P\left(\frac{1-A}{1-B}\right)$ for $|z|<r_0$, where r_0 is given by (3.1).

PROOF. We have $zf'(z)=g(z)p(z)$, $p\,\varepsilon\, P[A,B]$ and so

$$\frac{(zf'(z))'}{g'(z)}=p(z)+\frac{g(z)}{zg'(z)}\cdot zp'(z)$$

Thus

$$Re\left[\frac{(zf'(z))'}{g'(z)}-\frac{1-A}{1-B}\right]\ge Re\, p(z)\left[1-\frac{(1-Br)}{(1-Ar)}\cdot\frac{(A-B)r}{(1-Ar)(1-Br)}\right]-\frac{1-A}{1-B}$$

$$\ge\frac{(1-Ar)}{(1-Br)}\left[\frac{1-(3A-B)r+A^2r^2}{(1-Ar)(1-Br)}\right]-\frac{1-A}{1-B}$$

$$=\frac{(A-B)}{(1-B)}\left[\frac{1-(2+A-B)r+Ar^2}{(1-Ar)(1-Br)}\right]$$

$$f(z) = \frac{(S_1(z))^{\frac{k}{4}+\frac{1}{2}}}{(S_2(z))^{\frac{k}{4}-\frac{1}{2}}}, \qquad S_1, S_2 \,\varepsilon\, S^*[A,B]. \tag{1.2}$$

Clearly $k \geq 2$ and $R_2[A,B] = S^*[A,B]$. Also $R_k[1,-1] = U_k$, the class of functions with bounded radius rotation discussed in [3].

Similarly we can define the class $V_k[A,B]$ as follows. A function f, analytic in E and given by (1.1) belongs to $V_k[A,B]$, $k \geq 2$, if and only if

$$f'(z) = \frac{(S_1(z)/z)^{\frac{k}{4}+\frac{1}{2}}}{(S_2(z)/z)^{\frac{k}{4}-\frac{1}{2}}}, \qquad S_1, S_2 \,\varepsilon\, S^*[A,B] \tag{1.3}$$

From (1.2) and (1.3), it is clear that

$$f \,\varepsilon\, V_k[A,B] \quad \text{if and only if} \quad zf' \,\varepsilon\, R_k[A,B] \tag{1.4}$$

It may be noted that $V_2[A,B] = C[A,B]$ and $V_k[1,-1] = V_k$, the class of functions of bounded rotation first discussed by Paatero [4].

2. PRELIMINARY RESULTS

LEMMA 2.1 [5] Let p ε P[A,B]. Then

$$\frac{1-Ar}{1-Br} \leq Re\, p(z) \leq |p(z)| \leq \frac{1+Ar}{1+Br}$$

The following is the extension of Libera's result [6].

LEMMA 2.2. Let N and D be analytic in E, D map onto a many-sheeted starlike region. $N(0) = 0 = D(0)$ and $\frac{N'(z)}{D'(z)} \,\varepsilon\, P[A,B]$. Then $\frac{N(z)}{D(z)} \,\varepsilon\, P[A,B]$. For the proof of this result we refer to [5].

LEMMA 2.3. [7] Let $p \,\varepsilon\, P[A,B]$. Then, for $z \,\varepsilon\, E$, $\alpha \geq 0$ and $\beta \geq 0$, we have

$$Re\left\{\alpha p(z) + \beta \frac{zp'(z)}{p(z)}\right\} \geq \begin{cases} \dfrac{\alpha - \{\beta(A-B) + 2\alpha A\}r + \alpha A^2 r^2}{(1-Ar)(1-Br)}, & R_1 \leq R_2 \\[2ex] \beta\dfrac{A+B}{A-B} + \dfrac{2[(L_1K_1)^{1/2} - \beta(1-ABR^2)]}{(A-B)(1-r^2)}, & R_2 \leq R_1 \end{cases}$$

where

$$R_1 = \left(\frac{L_1}{K_1}\right)^{1/2}, \quad R_2 = \frac{1-Ar}{1-Br}, \quad L_1 = \beta(1-A)(1+Ar^2)$$

and

$$K_1 = \alpha(A-B)(1-r^2) + \beta(1-B)(1+Br^2).$$

This result is sharp.

3. MAIN RESULTS.

THEOREM 3.1. Let $f \,\varepsilon\, S^*[A,B]$. Then $f \,\varepsilon\, C\left(\frac{1-A}{1-B}\right)$ for

$$|z| < r_0 = \frac{2}{(2+A-B) + \sqrt{(2+A-B)^2 - 4A}} \tag{3.1}$$

This result is sharp.

PROOF. We have $zf'(z) = f(z)p(z)$, $p \,\varepsilon\, P[A,B]$

Hence $\frac{(zf'(z))'}{g'(z)} \in P\left(\frac{1-A}{1-B}\right)$ for $|z| < r_0$, where r_0 is given by (3.1).

Our next result is about the radius of convexity problem for the class $V_k[A,B]$.

THEOREM 3.4. Let $f \in V_k[A,B]$, $k \geq 2$. Then $f \in C\left(\frac{1-A}{1-B}\right)$ for $|z| < r_1$, where

$$r_1 = \frac{4}{k(1-B)+\sqrt{k^2(1-B)^2+16B}} \tag{3.2}$$

PROOF. Since $f \in V_k[A,B]$, we have from (1.3)

$$f'(z) = \frac{(S_1(z)/z)^{\frac{k}{4}+\frac{1}{2}}}{(S_2(z)/z)^{\frac{k}{4}-\frac{1}{2}}}, \qquad S_1, S_2 \in S^*[A,B]$$

This implies that

$$\frac{(zf'(z))'}{f'(z)} = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z), \; p_1, p_2 \in P[A,B]$$

so

$$Re\left[\frac{(zf'(z))'}{f'(z)} - \frac{1-A}{1-B}\right] \geq \left(\frac{k}{4}+\frac{1}{2}\right)\left(\frac{1-Ar}{1-Br}\right) - \left(\frac{k}{4}-\frac{1}{2}\right)\left(\frac{1+Ar}{1+Br}\right) - \frac{1-A}{1-B}$$

$$= \frac{(A-B)-\frac{k}{2}(1-B)(A-B)r - B(A-B)r^2}{(1-B)(1-B^2r^2)}$$

Hence $f \in C\left(\frac{1-A}{1-B}\right)$ for $|z| < r_1$, r_1 is given by (3.2).

From Theorem 3.4 and relation (1.4) we have the following:

THEOREM 3.5. Let $f \in R_k[A,B]$. Then $f \in S^*\left(\frac{1-A}{1-B}\right)$ for $|z| < r_1$ where r_1 is given by (3.2).

THEOREM 3.6. Let α and m be any positive integers and $f \in R_k[A,B]$. Then the function F defined by

$$(F(z))^\alpha = \frac{\alpha+m}{z^m}\int_0^z t^{m-1}(f(t))^\alpha dt \tag{3.3}$$

belongs to $S^*\left(\frac{1-A}{1-B}\right)$ for $|z| < r_1$, r_1 is given by (3.2).

PROOF. Let $J(z) = \int_0^z t^{m-1}(F(t))^\alpha dt$ and so

$$(F(z))^\alpha = \frac{\alpha+m}{z^m}J(z),$$

and

$$\frac{\alpha zF'(z)}{F(z)} = \frac{zJ'(z)}{J(z)} - m$$

or

$$\frac{zF'(z)}{F(z)} = \frac{1}{\alpha}\,\frac{zJ'(z)-mJ(z)}{J(z)} = \frac{N(z)}{D(z)}$$

$$N(0) = 0 = D(0)$$

By a result of Bernardi [8] and Theorem 3.5, $D(z)$ is a $(m+\alpha-1)$-valent starlike function for $|z|<r_1$. Also

$$\frac{N'(z)}{D'(z)}=\frac{1}{\alpha}\left\{\frac{(zJ'(z))'-mJ'(z)}{J'(z)}\right\}$$

$$=\frac{zf'(z)}{f(z)}$$

Now, by Theorem 3.5, $f\,\varepsilon\,S^*\left(\frac{1-A}{1-B}\right)$ for $|z|<r_1$ and this implies that $\frac{N'(z)}{D'(z)}\,\varepsilon\,P\left(\frac{1-A}{1-B}\right)$ for $|z|<r_1$. Hence

$$\frac{N(z)}{D(z)}\,\varepsilon\,P\left(\frac{1-A}{1-B}\right) \quad \text{for} \quad |z|<r_1, \quad \text{see [8].}$$

This proves our result.

Similarly, we can prove the following:

THEOREM 3.7. Let α and m be positive integers and $f\,\varepsilon\,V_k[A,B]$. Let F be defined by (3.3). Then $f\,\varepsilon\,C\left(\frac{1-A}{1-B}\right)$ for $|z|<r_1$ where r_1 is given by (3.2).

We now prove:

THEOREM 3.8. Let f and $g\,\varepsilon\,R_k[A,B]$ and, for α, m positive integers, let F be defined as

$$(F(z))^\alpha=\frac{(m+\alpha)}{(g(z))^m}\int_0^z t^{m-1}(f(t))^\alpha dt \tag{3.4}$$

Then $F\,\varepsilon\,S^*\left(\frac{1-A}{1-B}\right)$ for $|z|<r_0$

where $r_0=\min(r_1,r_2)$, r_1 is given by (3.2) and r_2 is the least positive root of the equation

$$\{(1-B)-\alpha(1-A)-\{(A-B(1+2m)\}r+\{(A-B)+2m(A-B)+\alpha(1-A)\}r^2=0\,, \tag{3.5}$$

PROOF. Let $J_1(z)=\frac{\alpha+m}{z^m}\int_0^z t^{m-1}(f(t))^\alpha dt$.

Then $(F(z))^\alpha=\left(\frac{z}{g(z)}\right)^m J_1(z)$, where by Theorem 3.6, $J_1\,\varepsilon\,S^*\left(\frac{1-A}{1-B}\right)$ for $|z|<r_1$.

So

$$\frac{\alpha zF'(z)}{F(z)}=\frac{zJ_1'(z)}{J_1(z)}+m\left(1-\frac{zg'(z)}{g(z)}\right)$$

Thus

$$Re\left[\frac{zF'(z)}{F(z)}-\frac{1-A}{1-B}\right]\geq\frac{1}{\alpha}\left[\left(1+\frac{B-A}{1-B}r\right)/(1+r)\right]+\left(\left[\frac{2m}{\alpha}(B-A)r\right]/(1-R)\right)-\frac{1-A}{1-B}$$

$$=\left\{\frac{\{(1-B-\alpha+\alpha A)+[(B-A)(1+2m)]r+[(A-B)+2m(A-B)+\alpha(1-A)]r^2\}}{\alpha(1-B)(1-r^2)}\right\}$$

This implies $Re\frac{zF'(z)}{F(z)}\geq\frac{1-A}{1-B}$ for $|z|<r_2$, where r_2 is the least positive root of (3.5). Hence $F\,\varepsilon\,S^*\left(\frac{1-A}{1-B}\right)$ for $|z|<r_0$, where $r_0=\min(r_1,r_2)$.

Similarly, we have the following:

THEOREM 3.9. Let f and $g\,\varepsilon\,V_k[A,B]$ and, for α, m positive integers, let F be defined by (3.4). Then $F\,\varepsilon\,C\left(\frac{1-A}{1-B}\right)$ for $|z|<r_0$, where r_0 is as given in Theorem 3.8.

THEOREM 3.10. Let $g\,\varepsilon\,V_k[A,B]$ and $\frac{f'(z)}{g'(z)}\,\varepsilon\,P[A,B]$ and let F be defined by

$$F(z)=\frac{m+1}{z^m}\int_0^z t^{m-1}f(t)dt\,,$$

where m is any positive integer. Then there exists a function G such that

$$\frac{F'(z)}{G'(z)}\,\varepsilon\,P\left(\frac{1-A}{1-B}\right),\quad G\,\varepsilon\,C\left(\frac{1-A}{1-B}\right)$$

for $|z|<r_1$, where r_1 is given by (3.2).

PROOF. Let

$$G(z)=\frac{m+1}{z^m}\int_0^z t^{m-1}g(t)dt\,.$$

Then, by Theorem 3.7 with $\alpha=1$, $G\,\varepsilon\,C\left(\frac{1-A}{1-B}\right)$ for $|z|<r_1$ and r_1 is defined by (3.2). Now

$$\frac{F'(z)}{G'(z)}=\frac{z^m f(z)-m\left(\int_0^z t^{m-1}f(t)dt\right)}{z^m q(z)-m\left(\int_0^z t^{m-1}g(t)dt\right)}$$

$$=\frac{\int_0^z t^m f'(t)dt}{\int_0^z t^m g'(t)dt}=\frac{N(z)}{D(z)}$$

Also

$$\frac{N'(z)}{D'(z)}=\frac{f'(z)}{g'(z)}\varepsilon P[A,B]\quad \text{for}\quad |z|<r_1$$

Thus, by Lemma 2.2, we have $\frac{N(z)}{D(z)}\varepsilon P[A,B]\subset P\left(\frac{1-A}{1-B}\right)$ for $|z|<r_1$ and this proves our result.

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