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ON THE THREE-DIMENSIONAL CR-SUBMANIFOLDS OF THE SIX-DIMENSIONAL SPHERE

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ABSTRACT. We show that the six-dimensional sphere does not admit three-dimensional totally umbilical proper CR-submanifolds.

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1. INTRODUCTION. The six-dimensional unit sphere $S^6(1)$ has a nearly Kaehler structure J constructed in a natural way by making use of Cayley division algebra [3]. It is because of this nearly Kaehler, non-Kaehler structure, that $S^6(1)$ has drawn the attention. In particular, almost complex submanifolds, CR-submanifolds and totally real submanifolds of $S^6(1)$ have been considered by A. Gray [4], K. Sekigawa and N. Ejiri [2]. For three-dimensional totally real submanifolds of $S^6(1)$ of constant curvature, N. Ejiri proved the following [2].

THEOREM 1. Let M be a 3-dimensional totally real submanifold of constant curvature c in $S^6(1)$. Then $c = 1$ (totally geodesic) or $c = \frac{1}{16}$ (minimal).

In this paper we consider 3-dimensional CR-submanifolds of $S^6(1)$. We prove the following result:

THEOREM 2. There are no 3-dimensional totally umbilical proper CR-submanifolds in $S^6(1)$.

2. PRELIMINARIES.

Let C_+ be the set of all purely imaginary Cayley numbers. The C_+ can be viewed as a 7-dimensional linear subspace $\mathbb{R}^7$ of $\mathbb{R}^8$. Consider the unit hypersphere which is centered at the origin

$$S^6(1) = \{x \varepsilon C_+ \mid < x, x > = 1\}.$$

The tangent space $T_x S^6$ of $S^6(1)$ at a point x may be identified with the affine subspace of C_+ which is orthogonal to x. On $S^6(1)$ define a (1,1)-tensor field J by putting

$$J_x U = x \times U,$$

where the above product is defined as in [3] for $x \varepsilon S^6(1)$ and $U \varepsilon T_x S^6$.

The above tensor field J determines an almost complex structure (i.e., $J^2 = -Id$) on $S^6(1)$. The compact simple lie group of automorphisms G_2 acts transitively on $S^6(1)$ and preserves both J and the standard metric on $S^6(1)$, [3].

Now let G be the (2,1)-tensor field on $S^6(1)$ defined by

$$G(X,Y) = (\bar{\nabla}_X J)Y$$

where $\bar{\nabla}$ is the Levi-Civita connection on $S^6(1)$ and $X, Y \varepsilon T_x S^6$.

Since $\bar{\nabla}_X J$ is skew-symmetric with respect to the Hermitian metric g on $S^6(1)$, it follows that G has the following property

$$g(G(X,Y),Z) + g(G(X,Z),Y) = 0 \tag{2.1}$$

where $X, Y, Z \varepsilon \mathfrak{X}(S^6)$.

A submanifold M of of $\dim(2p+q)$ in $S^6(1)$ is called a CR-submanifold if there exists a pair of orthogonal complementary distributions D and $\overset{\perp}{D}$ such that $JD = D$ and $J\overset{\perp}{D} \subset \nu$, where ν is the normal bundle of M and $\dim D = 2p$, $\dim \overset{\perp}{D} = q$[1]. Thus the normal bundle ν splits as $\nu = J\overset{\perp}{D} \oplus \mu$, where μ is invariant sub-bundle of ν under J.

A CR-submanifold is said to be proper if neither $D = \{0\}$ nor $\overset{\perp}{D} = \{0\}$.

We denote by $\nabla, \bar{\nabla}, \overset{\perp}{\nabla}$ the Riemannian connections on M, S^6 and the normal bundle respectively. They are related by Gauss formula and Weingarten formula:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.2}$$

$$\bar{\nabla}_X N = -A_N X + \overset{\perp}{\nabla}_X N \qquad N \varepsilon \nu \tag{2.3}$$

where $h(X,Y)$ and $A_N X$ are the second fundamental forms which are related by

$$g(h(X,Y),N) = g(A_N X, Y) \tag{2.4}$$

X and Y are vector fields on M.

Now a CR-submanifold is said to be totally umbilical if $h(X,Y) = g(X,Y)H$ where $H = \frac{1}{n}$ (trace h) is the mean curvature vector. If M is a totally umbilical CR-submanifold, then equations (2) and (3) become

$$\bar{\nabla}_X Y = \nabla_X Y + g(X,Y)H \tag{2.5}$$

$$\bar{\nabla}_X N = -g(H,N)X + \overset{\perp}{\nabla}_X N \tag{2.6}$$

Let R be the curvature tensor associated with ∇. Then the equation of Gauss is given by

$$R(X,Y;Z,W) = g(X,Z)g(Y,W) - g(Y,Z)g(X,W)$$

$$+ g(h(X,Z),h(Y,W)) - g(h(Y,Z),h(X,W))$$

It is known that for X, Y in D, $G(X,Y) = 0$, and $G(W,W) = 0$ for all $W \varepsilon \mathfrak{X}(S^6)$.

3. 3-DIMENSIONAL CR-SUBMANIFOLDS OF $S^6(1)$:

Let M be a 3-dimensional totally umbilical proper CR-submanifold of $S^6(1)$. Since M is proper, $D \neq \{0\}$ and $\overset{\perp}{D} \neq \{0\}$. Then since dim $M = 3$, we have $\dim D = 2$ and $\dim \overset{\perp}{D} = 1$.

We have the following:

LEMMA 1. If M is a 3-dimensional totally umbilical proper CR-submanifold of $S^6(1)$, then $H \varepsilon J\overset{\perp}{D}$.

PROOF. For $X, Y \neq 0$ in D we use equation (2.5) and the equation $J\bar{\nabla}_X Y = \bar{\nabla}_X JY$ to get

$$J\nabla_X Y + g(X,Y)JH = \nabla_X JY + g(X,JY)H. \tag{3.1}$$

Taking inner product in (3.1) with $N\varepsilon\mu$ we have

$$g(X,Y)g(JH,N) = g(X,JY)g(H,N) \tag{3.2}$$

In particular, if we let $Y = JX$ in (3.2) we get

$$\| X \| g(H,N) = 0$$

From which it follows that $H\varepsilon J\overset{\perp}{D}$.

LEMMA 2. If M is a 3-dimensional totally umbilical CR-submanifold of $S^6(1)$, the $\| H \|$ is constant.

PROOF. Using (2.7) and the equation $h(X,Y) = g(X,Y)H$ we get

$$R(X,Y;Z,W) = (1 + \| H \|^2)\{g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\} \tag{3.3}$$

Then since $\dim M = 3$, we invoke Schur's theorem to conclude that $(1 + \| H \|^2)$ is constant. Thus $\| H \|$ is constant.

4. PROOF OF THEOREM 2.

In this section let $\{X, JX, Z\}$ denote an orthonormal frame field for the 3-dimensional totally umbilical CR-submanifold M of $S^6(1)$. The unit vector fields X, JX are in D and the unit vector field Z is in $\overset{\perp}{D}$. Since M is totally umbilical, the equation $h(X,Y) = g(X,Y)H$ implies that

$$h(X,JX) = h(X,Z) = h(JX,Z) = 0$$

and

$$h(X,X) = h(JX,JX) = h(Z,Z) = H \tag{4.1}$$

We know from the previous Lemma that $H\varepsilon J\overset{\perp}{D}$. Since $\dim J\overset{\perp}{D} = 1$, then one can write $H = \alpha JZ$ for some smooth function α on M. Therefore

$$h(X,X) = h(JX,JX) = h(Z,Z) = \alpha JZ$$

Using equation (2.4) with $N = JZ$ we get

$$A_{JZ}X = \alpha X, \qquad A_{JZ}JX = \alpha JX, \qquad A_{JZ}Z = \alpha Z \tag{4.2}$$

So the frame field $\{X, JX, Z\}$ diagonalizes A. Now in $S^6(1)$ we have equation (2.1) i.e. $g((\bar{\nabla}_X J)Y, Z) + g((\bar{\nabla}_X J)Z, Y) = 0$ for any $X, Y, Z\varepsilon\mathfrak{X}(S^6)$. Since for $X, Y\varepsilon D$ $(\bar{\nabla}_X J)Y = 0$, then using this equation with $Y = JX$ for our orthonormal frame field $\{X, JX, Z\}$ in M, we get

$$g((\bar{\nabla}_X J)Z, JX) = 0 \tag{4.3}$$

Using equation (2.5), (4.3) and (2.6) with the fact that $H\varepsilon J\overset{\perp}{D}$ and $(\bar{\nabla}_X J)Z = \bar{\nabla}_X JZ - J\bar{\nabla}_X Z$ we get

$$g(\nabla_X Z, X) = 0 \tag{4.4}$$

Again using equation (2.5) and (2.6) in equation (2.1) with $Y = X$, we get

$$g(\nabla_X Z, JX) = \alpha \tag{4.5}$$

Also using equation (2.1) and $(\bar{\nabla}_{JX}J)Z = \bar{\nabla}_{JX}JZ - J\ \bar{\nabla}_{JX}Z$ we get

$$g(\nabla_{JX}Z, X) = -\alpha \tag{4.6}$$

Switching the role of X and Y in equation (2.1) and letting $Y = JX$ we obtain

$$g(\nabla_{JX}Z, JX) = 0 \tag{4.7}$$

Now using the equation $g((\bar{\nabla}_X J)X, JZ) = 0$ and $g(\bar{\nabla}_{JX}J)x, z) = 0$ we get

$$g(\nabla_X X, Z) = 0, \qquad g(\nabla_{JX}JX, Z) = 0 \tag{4.8}$$

From the equation $(\bar{\nabla}_Z J)Z = 0$, using equation (4.1) and (4.2) and the fact that $\nabla_Z Z \varepsilon D$, we get

$$\nabla_Z Z = 0, \qquad \overset{\perp}{\nabla}_Z JZ = 0 \tag{4.9}$$

Using equations (4.4), (4.5), (4.6), (4.7), (4.8) and the first part of equation (4.9) we can write the local equations for the frame field $\{X, JX, Z\}$ as follows:

$$\nabla_X Z = \alpha JX, \qquad \nabla_{JX}Z = -\alpha X, \qquad \nabla_Z Z = 0$$

$$\nabla_X X = aJX, \qquad \nabla_{JX}X = -bJX + \alpha Z, \qquad \nabla_Z X = cJX$$

$$\nabla_X JX = -aX - \alpha Z, \qquad \nabla_{JX}JX = bX, \qquad \nabla_Z JX = -cX \tag{4.10}$$

for some smooth functions a, b and c.

The curvature tensor R is given by

$$R(X, Y; Z, W) = \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W \rangle$$

Then using this equation with the help of equations (4.10) we get $R(X, Z, Z, X) = \alpha^2$, $\alpha = \| H \|$. But from equation (3.3) we know that $R(X, Z, Z, X) = -(1 + \alpha^2)$. This is a contradiction and hence $S^6(1)$ cannot admit a 3-dimensional totally umbilical proper CR-submanifolds.

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