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ALMOST COMPLEX SURFACES IN THE NEARLY KAEHLER S^6

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ABSTRACT: It is shown that a compact almost complex surface in S^6 is either totally geodesic or the minimum of its Gaussian curvature is less than or equal to 1/3.

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1. INTRODUCTION.

The six dimensional sphere S^6 has almost complex structure J which is nearly Kaehler, that is, it satisfies $(\bar{\nabla}_X J)(X) = 0$, where $\bar{\nabla}$ is the Riemannian connection on S^6 corresponding to the usual metric g on S^6. Sekigawa [1] has studied almost complex surfaces in S^6 and has shown that if they have constant curvature K, then either $K = 0$, $1/6$ or 1. Under the assumption that the almost complex surface M in S^6 is compact, he has shown that if $K > 1/6$, then $K = 1$ and if $1/6 \leq K < 1$, then $K = 1/6$. Dillen et al [2-3] have improved this result by showing if $1/6 \leq K \leq 1$, then either $K = 1/6$ or $K = 1$ and if $0 \leq K \leq 1/6$, then either $K = 0$ or $K = 1/6$. However, using system of differential equations (1) (cf. [5], p. 67) one can construct examples of almost complex surfaces in S^6 whose Gaussian curvature takes values outside $[9, 1/6]$ or $[1/6, 1]$. The object of the present paper is to prove the following:

THEOREM 1. Let M be a compact almost complex surface in S^6 and K_0 be the minimum of the Gaussian curvature of M. Then either M is totally geodesic or $K_0 \leq 1/3$.

2. MAIN RESULTS. Let M be a 2-dimensional complex submanifold of S^6 and g be the induced metric on M. The Riemannian connection $\bar{\nabla}$ of S^6 induces the Riemannian connection ∇ on M and the connection $\nabla^{\perp}$ in the normal bundle ν. We have the Gauss and Weingarten formulae

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \quad X, Y \in \mathfrak{X}(M), \quad N \in \nu, \tag{2.1}$$

where h, A_N are the second fundamental forms satisfying $g(h(X, Y), N) = g(A_N X, Y)$ and $\mathfrak{X}(M)$ is the Lie-algebra of vector fields on M. The curvature tensors $\bar{R}$, R and $R^{\perp}$ of the connections $\bar{\nabla}$,

∇ and $\nabla^{\perp}$ respectively satisfy

$$R(X,Y;Z,W) = \bar{R}(X,Y;Z,W) + g(h(Y,Z),h(X,W)) - g(h(X,Z),h(Y,W)) \tag{2.2}$$

$$\bar{R}(X,Y;N_1,N_2) = R^{\perp}(X,Y:N_1,N_2) - g([A_{N_1},A_{N_2}](X),Y) \tag{2.3}$$

$$[\bar{R}(X,Y)Z]^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z), \quad X,Y,Z,W \varepsilon \mathfrak{X}(M),\ N_1,N_2 \varepsilon \nu, \tag{2.4}$$

where $[\bar{R}(X,Y)Z]^{\perp}$ is the normal component of $\bar{R}(X,Y)Z$, and

$$(\bar{\nabla}_X h)(Y,Z) = \nabla_X^{\perp} h(Y,Z) - h(Y, \nabla_X Z).$$

The curvature tensor $\bar{R}$ of S^6 is given by

$$\bar{R}(X,Y;Z,W) = g(Y,Z)g(X,W) - g(X,Z)g(Y,W). \tag{2.5}$$

LEMMA 1. Let M be a 2-dimensional complex submanifold of S^6. Then $(\bar{\nabla}_X J)(Y) = 0$, $X, Y \varepsilon \mathfrak{X}(M)$.

PROOF. Take a unit vector field $X \varepsilon \mathfrak{X}(M)$. Then $\{X, JX\}$ is orthonormal frame on M. Since S^6 is nearly Kaehler manifold we have $(\bar{\nabla}_X J)(X) = 0$, and $(\bar{\nabla}_{JX} J)(JX) = 0$. Also

$$(\bar{\nabla}_X J)(JX) = -J(\bar{\nabla}_X J)(X) = 0 \text{ and } (\bar{\nabla}_{JX} J)(X) = -(\bar{\nabla}_X J)(JX) = 0.$$

Now for any $Y, Z \varepsilon \mathfrak{X}(M)$, we have $Y = aX + bJX$ and $Z = cX + dJX$, where a, b, c and d are smooth functions. We have

$$(\bar{\nabla}_Y J)(Z) = a(\bar{\nabla}_X J)(Z) + b(\bar{\nabla}_{JX} J)(Z) = -a(\bar{\nabla}_Z J)(X) - b(\bar{\nabla}_Z J)(JX)$$
$$= -ac(\bar{\nabla}_X J)(X) - ad(\bar{\nabla}_{JX} J)(X) - bc(\bar{\nabla}_X J)(JX) - bd(\bar{\nabla}_{JX} J)(JX) = 0.$$

LEMMA 2. For a 2-dimensional complex submanifold M of S^6, the following hold

(i) $h(X, JY) = h(JX, Y) = Jh(X,Y)$, $\quad \nabla_X JY = J\nabla_X Y$,

(ii) $JA_N X = A_{JN} X$, $A_N JX = -JA_N X$,

(iii) $(\bar{\nabla}_{JX} h)(Y,Z) = (\bar{\nabla}_X h)(JY,Z) = (\bar{\nabla}_X h)(Y,JZ)$,

(iv) $R(X,Y)JZ = JR(X,Y)Z$, $\quad X, Y, Z \varepsilon \mathfrak{X}(M), N \varepsilon \nu$.

PROOF. (i) follows directly from Lemma 1 and equation (2.1). The second part of (ii) follows from (i). For first part of (ii), observe that for $N \varepsilon \nu$ and $X \varepsilon \mathfrak{X}(M)$, $g((\bar{\nabla}_X J)(N), Y) = -g(N, (\bar{\nabla}_X J)(Y)) = 0$ for each $Y \varepsilon \mathfrak{X}(M)$, that is, $(\nabla_X J)(N)$ is normal to M. Hence expanding $(\bar{\nabla}_X J)(N)$ using (2.1) and equating the tangential parts we get the first part of (ii).

From equations (2.4) and (2.5), we get

$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z) = (\bar{\nabla}_Z h)(X,Y), \quad X,Y,Z \varepsilon \mathfrak{X}(M). \tag{2.6}$$

Also from (i) we have

$$(\bar{\nabla}_X h)(JY,Z) = (\bar{\nabla}_X h)(Y,JZ), \quad X, Y \varepsilon \mathfrak{X}(M). \tag{2.7}$$

Thus from (2.6) and (2.7), we get that

$$(\bar{\nabla}_X h)(JY,Z) = (\bar{\nabla}_X h)(Y,JZ) = (\bar{\nabla}_Y h)(X,JZ) = (\bar{\nabla}_Y h)(JX,Z) = (\bar{\nabla}_{JX} h)(Y,Z),$$

this together with (2.7) proves (iii). The proof of (iv) follows from second part of (i).

The second covariant derivative of the second fundamental form is defined as

$$(\bar{\nabla}^2 h)(X,Y,Z,W) = \nabla^{\perp}{}_X(\bar{\nabla} h)(Y,Z,W) - (\bar{\nabla} h)(\nabla_X Y, Z, W) - (\bar{\nabla} h)(Y, \nabla_X Z, W) - (\bar{\nabla} h)(Y, Z, \nabla_X W),$$

where $(\bar{\nabla} h)(X,Y,Z) = (\bar{\nabla}_X h)(Y,Z)$, $\quad X, Y, Z, W \varepsilon \mathfrak{X}(M)$.

Let $\Pi: UM \to M$ and UM_p be the unit tangent bundle of M and its fiber over $p\varepsilon M$ respectively. Define the function $f: UM \to R$ by $f(U) = \| h(U,U) \|^2$.

For $U\varepsilon UM_p$, let $\sigma_U(t)$ be the geodesic in M given by the initial conditions $\sigma_U(0) = p$, $\dot{\sigma}_U(0) = U$. By parallel translating a $V\varepsilon UM_p$ along $\sigma_U(t)$, we obtain a vector field $V_U(t)$. We have the following Lemma (cf. [5]).

LEMMA 3. For the function $f_U(t) = f(V_U(t))$, we have

(i) $\frac{d}{dt} f_U(t) = 2g((\bar{\nabla} h)(\dot{\sigma}_U, V_U, V_U), h(V_U, V_U))(t)$,

(ii) $\frac{d^2}{dt^2} f_U(0) = 2g((\bar{\nabla}^2)(U,U,V,V), h(V,V)) + 2\| (\bar{\nabla} h)(U,V,V) \|^2$.

3. PROOF OF THE THEOREM 1. Since UM is compact, the function f attains maximum at some $V\varepsilon UM$. From (i) of Lemma 2, $\| h(V,V) \|^2 = \| h(JV,JV) \|^2$ and thus we have $\frac{d^2}{dt^2} f_V(0) \leq 0$ and $\frac{d^2}{dt^2} f_{JV}(0) \leq 0$. Using (iii) of Lemma 2 in (2.8) we get that

$$(\bar{\nabla}^2 h)(JV, JV, V, V) = (\bar{\nabla}^2 h)(JV, V, JV, V).$$

The above equation together with the Ricci identity gives

$$\begin{aligned}(\bar{\nabla}^2 h)(JV, JV, V, V) &- (\bar{\nabla}^2 h)(JV, V, JV, V).\\ &= (\bar{\nabla}^2 h)(JV, V, JV, V) - (\bar{\nabla}^2 h)(V, JV, JV, V)\\ &= R^{\perp}(JV, V) h(JV, V) - h(R(JV,V)JV, V) - h(JV, R(JV,V)V).\end{aligned}$$

Taking inner product with $h(V,V)$ and using (iv) of Lemma 2, we get

$$\begin{aligned}g((\bar{\nabla}^2 h)(JV, JV, V, V) &- (\bar{\nabla}^2 h)(V, JV, JV, V), h(V,V))\\ &= R^{\perp}(JV, V; h(JV,V), h(V,V)) - 2g(h(R(JV,V)JV, V), h(V,V)).\end{aligned} \tag{3.1}$$

Now using (i) of Lemma 2, we find that $g(h(U,U), h(U,JU)) = 0$, that is, $g(A_{h(U,}U_{U)}, JU) = 0$ for all $U\varepsilon UM_p$. Since $\dim M = 2$, it follows that $A_{h(U,}U_{U)} = \lambda U$. To find λ, we take inner inner product with U and obtain $\lambda = \| h(U,U) \|^2$. Thus, $A_h U_{(U,U)} = \| h(U,U) \|^2 U$. From equations (2.2) and (2.5) we obtain

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y + A_{h(Y,Z)}X - A_{h(X,Z)}Y,$$

which gives

$$R(JV,V)JV = -V + A_{h(V,}JV_{JV)} - A_{h(JV,}V_{JV)} = -V + 2A_{h(V,}V_{V)} = -V + 2\| h(V,V) \|^2 V. \tag{3.2}$$

Also from (2.3) and (2.5) we get

$$\begin{aligned}R^{\perp}(JV, V, h(JV,V), h(V,V)) &= g([A_{h(JV,V)}, A_{h(V,V)}](JV), V)\\ &= -2g(A_{h(V,V)}V, A_{h(V,V}V_{)})\\ &= -2\| h(V,V) \|^4.\end{aligned}$$

Substituting (3.2) and (3.3) in (3.1) we get

$$g((\bar{\nabla}^2 h)(JV, JV, V, V) - (\bar{\nabla}^2 h)(V, JV, JV, V), h(V, V)) = 2f(V)(1 - 3f(V)). \quad (3.4)$$

From (iii) of Lemma 2, it follows that

$$(\bar{\nabla} h)(JV, JV, V) = (\bar{\nabla} h)(J^2V, V, V) = -(\bar{\nabla} h)(V, V, V),$$

this together with $\nabla_X JY = J \nabla_X Y$ of (i) in Lemma 2, gives

$$(\nabla^2 h)(V, JV, JV, V) = -(\bar{\nabla}^2 h)(V, V, V, V).$$

Using this and (ii) of Lemma 3 in (3.4), we obtain

$$\frac{d^2}{dt^2} f_V(0) + \frac{d^2}{dt^2} f_{JV}(0) = 2f(V)(1 - 3f(V)) + 2 \| (\bar{\nabla} h)(V, V, V) \|^2 + 2 \| (\bar{\nabla} h)(JV, V, V) \|^2 \leq 0$$

Thus either $f(V) = 0$, that is, M is totally geodesic or $1/3 \leq f(V)$. Since an orthonormal frame of M is of the form (U, JU), the Gaussian curvature K of M is given by

$$K = 1 + g(h(U, U), h(JU, JU)) - g(h(U, JU), h(U, JU)) = 1 - 2 \| h(U, U) \|^2.$$

Thus $K: UM \to R$, is a smooth function, and UM being compact, K attains its minimum $K_0 = \min K$ and we have $K_0 = 1 - 2\max \| h(U, U) \|^2$, from which for the case $1/3 \leq f(V)$, we get $K_0 \leq 1/3$. This completes the proof of the Theorem.

As a direct consequence of our Theorem we have

COROLLARY. Let M be a compact almost complex surface in S^6. If the Gaussian curvature K of M satisfies $K > 1/3$, then M is totally geodesic.

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