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HODOGRAPH METHOD IN MHD ORTHOGONAL FLUID FLOWS

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Abstract. Equations for steady plane MHD orthogonal flows of a viscous incompressible fluid of finite electrical conductivity are recast in the hodograph plane by using the Legendre transform function of the streamfuntion. Three examples are studied to illustrate the developed theory. Solutions and geometries for these examples are determined.

Key Words and Phrases: Steady. Hodographic study. orthogonal. Magnetohydrodynamic. electrically conducting. finite conductivity.
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1. Introduction. This paper deals with the application of the hodograph transformation for solving a system of non-linear partial differential equations governing steady plane magnetohydrodynamic flow of a viscous incompressible fluid in the presence of a magnetic field. W. F. Ames [1] has given an excellent survey to this method together with its applications to various other fields. Recently. O. P. Chandna et al. [2.3] used the hodograph and Legendre transformations to study non-Newtonian steady plane aligned and transverse MHD flows. O. P. Chandna et al. [4] also applied this technique to Navier Stokes equations. In this paper we consider the magnetic and velocity field vector are mutually orthogonal and the electrical conductivity of the fluid is taken to be finite. Since electrical conductivity is finite for most viscous fluid. our accounting for finite electrical conductivity makes the flow problem realistic and attractive from both a mathematic and physical point of view. We study our flows with the objective of determining exact solutions to various flow configurations. The plan of this paper is as follows: In section 2 the equations are cast into a convenient form for this work. Section 3 contains the transformation of equations to the hodograph plane so that the role of independent variables x. y and the dependent variables u. v are interchanged. In section 4 we introduce a Legendre transform function of the streamfunction and obtain a system of three equations in the Legendre transform function and the proportionality function. In section 5. we demonstrate the use of theoretical results found in section 4 by determining solutions to the following flows: (a) vortex flows (b) radial flows (c) spiral flows

2. Basic equations. The steady. plane flow of a viscous. incompressible fluid of finite electrical conductivity is governed by the following system of equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2.1}$$

$$\zeta\left(u\frac{\partial u}{\partial y}+v\frac{\partial u}{\partial y}\right)+\frac{\partial p}{\partial x}=\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\mu^* jH_2 \tag{2.2}$$

$$\zeta\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)+\frac{\partial p}{\partial y}=\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\mu^* jH_1 \tag{2.3}$$

$$uH_2-vH_1=\frac{1}{\mu^*\sigma}j+K \tag{2.4}$$

$$\frac{\partial H_1}{\partial x}+\frac{\partial H_2}{\partial y}=0 \tag{2.5}$$

$$j=\frac{\partial H_2}{\partial x}-\frac{\partial H_1}{\partial y} \tag{2.6}$$

where $u.v$ are the components of the velocity field $\mathbf{V}.H_1.H_2$ are the components of the magnetic field $\mathbf{H}$. p the pressure. ζ the constant fluid density. μ the constant coefficient of viscosity and μ^* the constant magnetic permeability. Here K is an arbitrary constant of integration obtained from the diffusion equation

$$\operatorname{curl}\left[\mathbf{V}\times\mathbf{H}-\frac{1}{\mu^*\sigma}\operatorname{curl}\mathbf{H}\right]=\mathbf{0}$$

Introducing the functions

$$\omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}.h=\frac{1}{2}\zeta|\mathbf{V}|^2+p \tag{2.7}$$

where

$$|\mathbf{V}|^2=u^2+v^2$$

to the above system of equations. we obtain the following system:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{8}$$

$$\frac{\partial h}{\partial x}=\zeta v\omega-\mu\omega_y-\mu^*jH_2 \tag{2.8}$$

$$\frac{\partial h}{\partial y}=-\zeta u\omega+\mu\omega_x+\mu^*jH_1 \tag{2.9}$$

$$uH_2-vH_1=\frac{1}{\mu^*\sigma}j+K \tag{11}$$

$$\frac{\partial H_1}{\partial x}+\frac{\partial H_2}{\partial y}=0 \tag{2.10}$$

$$j=\frac{\partial H_2}{\partial x}-\frac{\partial H_1}{\partial y} \tag{2.11}$$

$$\omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} \tag{2.12}$$

of seven partial differential equations in seven unknowns $u.v.H_1.H_2.\omega.j$ and h as functions of $x.y$.

We consider our flows to be orthogonal flows. A plane flow is said to be orthogonal when the velocity field vector and the magnetic field vector are mutually perpendicular in the flow region. From this definition. we have

$$\mathbf{H}=\mathbf{k}\times f(x.y)\mathbf{V} \tag{2.15}$$

where $\mathbf{k}=(0.0.1)$ and f is a scalar function.

Using (2.15) in the system of equations (2.8) to (2.14). we get

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2.16}$$

$$\frac{\partial h}{\partial x}=\zeta vw-\mu\omega_y-\mu^*jfu \tag{2.17}$$

$$\frac{\partial h}{\partial y}=-\zeta uw+\mu\omega_x-\mu^*jfv \tag{2.18}$$

$$f(u^2+v^2)=\frac{1}{\mu^*\sigma}j+K \tag{2.19}$$

$$-f\omega+u\frac{\partial f}{\partial y}-v\frac{\partial f}{\partial x}=0 \tag{2.20}$$

$$j = u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} \tag{2.21}$$

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{2.22}$$

3. Equations in the Hodograph Plane. Letting the flow variables $u = u(x.y).v = v(x.y)$ to be such that. in the region of flow under consideration. the Jacobian

$$J = \frac{\partial(u.v)}{\partial(x.y)} = u_x v_y - u_y v_x \neq 0. |J| < \infty$$

we may consider $x.y$ as functions of $u.v$ By means of

$$x = x(u.v). \qquad y = y(u.v)$$

we derive the following relations

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= J\frac{\partial y}{\partial v}. & \frac{\partial u}{\partial y} &= -J\frac{\partial x}{\partial v} \\ \frac{\partial v}{\partial x} &= -J\frac{\partial y}{\partial u}. & \frac{\partial v}{\partial y} &= J\frac{\partial x}{\partial u} \end{aligned}\right\} \tag{3.1}$$

$$J(x.y) = \frac{\partial(u.v)}{\partial(x.y)} = \left[\frac{\partial(x.y)}{\partial(u.v)}\right]^{-1} = \bar{J}(u.v) \tag{3.2}$$

$$\left.\begin{aligned} \frac{\partial g}{\partial x} &= \frac{\partial(g.y)}{\partial(x.y)} = J\frac{\partial(\bar{g}.y)}{\partial(u.v)} = \bar{J}\frac{\partial(\bar{g}.y)}{\partial(u.v)} \\ \frac{\partial g}{\partial x} &= -\frac{\partial(g.x)}{\partial(x.y)} = J\frac{\partial(x.\bar{g})}{\partial(u.v)} = \bar{J}\frac{\partial(x.\bar{g})}{\partial(u.v)} \end{aligned}\right\} \tag{3.3}$$

where $g = g(x.y) = g(x(u.v).y(u.v)) = \bar{g}(u.v)$ is any continuously differentiable function.

Employing these transformation relations for the first order partial derivatives appearing in the system of equations (2.16) to (2.22). weobtain the system of equations in the $(u.v)$ - plane as follows:

$$\frac{\partial x}{\partial u} + \frac{\partial y}{\partial v} = 0 \tag{3.4}$$

$$\bar{J}\frac{\partial(\bar{h}.y)}{\partial(u.v)} = \zeta v\bar{\omega} - \mu\bar{J}W_1 - \mu^*\bar{j}\bar{f}u \tag{3.5}$$

$$\bar{J}\frac{\partial(x.\bar{h})}{\partial(u.v)} = -\zeta u\bar{\omega} + \mu\bar{J}W_2 - \mu^*\bar{j}\bar{f}v \tag{3.6}$$

$$\bar{f}(u^2 + v^2) = \frac{1}{\mu^*\sigma}\bar{j} + K \tag{3.7}$$

$$-\bar{f}\bar{\omega} + \bar{J}\left[u\frac{\partial(x.\bar{f})}{\partial(u.v)} - v\frac{\partial(\bar{f}.y)}{\partial(u.v)}\right] = 0 \tag{3.8}$$

$$\bar{j} = \bar{J}\left[u\frac{\partial(\bar{f}.y)}{\partial(u.v)} + v\frac{\partial(x.\bar{f})}{\partial(u.v)}\right] \tag{3.9}$$

$$\bar{\omega} = \bar{J}\left[\frac{\partial x}{\partial v} - \frac{\partial y}{\partial u}\right] \tag{3.10}$$

where

$$\bar{J} = \left[\frac{\partial(x.y)}{\partial(u.v)}\right]^{-1}.W_1 = \frac{\partial(x.\bar{\omega})}{\partial(u.v)}.W_2 = \frac{\partial(\bar{\omega}.y)}{\partial(u.v)} \tag{3.11}$$

This is a system of seven partial differential equations in six unknown functions $x = x(u.v).y = y(u.v).\bar{\omega} = \bar{\omega}(u.v).\bar{f} = \bar{f}(u.v).\bar{j} = \bar{j}(u.v)$ and $\bar{h} = \bar{h}(u.v)$ and an arbitrary constant K. Once $x.y.\bar{\omega}.\bar{f}.\bar{j}$ and $\bar{h}$ are determined. we can find $u = u(x.y).v = v(x.y).\omega = \omega(x.y).f = f(x.y).j = j(x.y)$ and $h = h(x.y)$ which are the solutions for the system of equations (2.16) to (2.22).

Equations in Legendre transform function and $\bar{f}(u,v)$.

The equation of continuity implies the existence of a streamfunction $\psi(x,y)$, so that

$$d\psi = -vdx + udy \text{ or } \frac{\partial\psi}{\partial x} = -v, \quad \frac{\partial\psi}{\partial y} = u \tag{3.12}$$

Likewise, equation (3.4) implies the existence of a function $L(u,v)$, called the Legendre transform function of the streamfunction $\psi(x,y)$, so that

$$dL = -ydu + xdv \text{ or } \frac{\partial L}{\partial u} = -y, \frac{\partial L}{\partial v} = x \tag{3.13}$$

and the two functions $\psi(x,y)$ and $L(u,v)$ are related by

$$L(u,v) = vx - uy + \psi(x,y) \tag{3.14}$$

Introducing $L(u,v)$ into the system of equations (3.4) to (3.10), we obtain

$$\bar{J}\frac{\partial(\frac{\partial L}{\partial u},\bar{h})}{\partial(u,v)} = \zeta v\bar{\omega} - \mu\bar{J}W_1 - \mu^*\bar{j}\bar{f}u \tag{3.15}$$

$$\bar{J}\frac{\partial(\frac{\partial L}{\partial v},\bar{h})}{\partial(u,v)} = -\zeta u\bar{\omega} + \mu\bar{J}W_2 - \mu^*\bar{j}\bar{f}v \tag{3.16}$$

$$\bar{f}(u^2+v^2) = \frac{1}{\mu^*\sigma}\bar{j} + K \tag{3.17}$$

$$-\bar{f}\bar{\omega} + \bar{J}\left[u\frac{\partial(\frac{\partial L}{\partial v},\bar{f})}{\partial(u,v)} - v\frac{\partial(\frac{\partial L}{\partial u},\bar{f})}{\partial(u,v)}\right] = 0 \tag{3.18}$$

$$\bar{j} = \bar{J}\left[u\frac{\partial(\frac{\partial L}{\partial u},\bar{f})}{\partial(u,v)} + v\frac{\partial(\frac{\partial L}{\partial v},\bar{f})}{\partial(u,v)}\right] \tag{3.19}$$

$$\bar{\omega} = \bar{J}\left[\frac{\partial^2 L}{\partial v^2} + \frac{\partial^2 L}{\partial u^2}\right] \tag{3.20}$$

where

$$\bar{J} = \left[\frac{\partial^2 L}{\partial v^2}\frac{\partial^2 L}{\partial u^2} - \left(\frac{\partial^2 L}{\partial u\partial v}\right)^2\right]^{-1} \tag{3.21}$$

$$W_1 = \frac{\partial(\frac{\partial L}{\partial v},\bar{\omega})}{\partial(u,v)}, W_2 = \frac{\partial(\frac{\partial L}{\partial u},\bar{\omega})}{\partial(u,v)} \tag{3.22}$$

By using the integrability condition

$$\left[\bar{J}\frac{\partial^2 L}{\partial u\partial v}\frac{\partial}{\partial v} - \bar{J}\frac{\partial^2 L}{\partial v^2}\frac{\partial}{\partial u}\right]\left(\bar{J}\frac{\partial(\frac{\partial L}{\partial u},\bar{h})}{\partial(u,v}\right)$$

$$= \left[\bar{J}\frac{\partial^2 L}{\partial u^2}\frac{\partial}{\partial v} - \bar{J}\frac{\partial^2 L}{\partial u\partial v}\frac{\partial}{\partial u}\right]\left(\bar{J}\frac{\partial(\frac{\partial L}{\partial v},\bar{h})}{\partial(u,v)}\right)$$

we eliminate $\bar{h}(u,v)$ from equations (3.15) and (3.16) to obtain

$$\zeta(uW_2 + vW_1) - \mu\left[\frac{\partial\left(\frac{\partial L}{\partial v},\bar{J}W_1\right)}{\partial(u,v)} + \frac{\partial\left(\frac{\partial L}{\partial u},\bar{J}W_2\right)}{\partial(u,v)}\right]$$

$$+\mu^*\left[v\frac{\partial\left(\frac{\partial L}{\partial u},\bar{f}\bar{j}\right)}{\partial(u,v)} - u\frac{\partial\left(\frac{\partial L}{\partial v},\bar{f}\bar{j}\right)}{\partial(u,v)} + \bar{f}\bar{j}\left(\frac{\partial^2 L}{\partial u^2} + \frac{\partial^2 L}{\partial v^2}\right)\right] = 0 \tag{3.23}$$

Employing (3.17) in (3.23) and making use of (3.18), we get

$$\zeta(uW_2 + vW_1) - \mu \left[\frac{\partial\left(\frac{\partial L}{\partial v}.\bar{J}W_1\right)}{\partial(u.v)} + \frac{\partial\left(\frac{\partial L}{\partial u}.\bar{J}W_2\right)}{\partial(u.v)} \right]$$

$$+\mu^{*^2}\sigma\bar{f}\left[v\frac{\partial\left(\frac{\partial L}{\partial u}.(u^2+v^2)\bar{f}\right)}{\partial(u.v)} - u\frac{\partial\left(\frac{\partial L}{\partial v}.(u^2+v^2\bar{f})\right)}{\partial(u.v)} \right] = 0 \tag{3.24}$$

From (3.17) and (3.19), we obtain

$$\mu^*\sigma(u^2+v^2)\bar{f} - \bar{J}\left[u\frac{\partial\left(\frac{\partial L}{\partial u}.\bar{f}\right)}{\partial(u.v)} + v\frac{\partial\left(\frac{\partial L}{\partial v}.\bar{f}\right)}{\partial(u.v)} \right] = \mu^*\sigma K \tag{3.25}$$

In summary, we have the following theorem:

Theorem 1. If $L(u.v)$ is the Legendre transform function of a streamfunction of steady plane orthogonal flow of an incompressible viscous fluid of finite electrical conductivity and $\bar{f}(u.v)$is the transformed proportionality function, then $L(u.v)$ and $\bar{f}(u.v)$ must satisfy equations (3.18), (3.24) and (3.25) where $\bar{\omega}$, $\bar{J}$, $W_1.W_2$ and $\bar{j}$ are given by (3.10), (3.21), (3.22) and (3.17).

Once a solution $\{L(u.v), \bar{f}(u.v)\}$ is found, for which $\bar{J}$ evaluated from (3.21) satisfies $0 < |J| < \infty$, the solution for the velocity components are obtained by solving equations (3.13) simultaneously. Having obtained the velocity components $u = u(x.y)$, $v = v(x.y)$, we obtain $f(x.y)$ in the physical plane from the solution for $\bar{f}(u.v)$ in the hodograph plane. Using $\mathbf{V}(x.y)$ and $f(x.y)$ in (2.6), (2.7), (2.9) and (2.10), we determine other flow variables in the physical plane.

We now develop the results of the above theorem in polar coordinates $(q.\theta)$ in the hodograph plane. Defining

$$q^2 = u^2 + v^2, \theta = \tan^{-1}\left(\frac{v}{u}\right),$$

we have the following transformation relations

$$\frac{\partial}{\partial u} = \cos\theta\frac{\partial}{\partial q} - \frac{\sin\theta}{q}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial v} = \sin\theta\frac{\partial}{\partial q} + \frac{\cos\theta}{q}\frac{\partial}{\partial\theta}$$

$$\frac{\partial(F.G)}{\partial(u.v)} = \frac{\partial(F^*.G^*)}{\partial(q.\theta)}\frac{\partial(q.\theta)}{\partial(u.v)} = \frac{1}{q}\frac{\partial(F^*.G^*)}{\partial(q.\theta)}$$

where $F(u.v) = F^*(q.\theta)$, $G(u.v) = G^*(q.\theta)$ are continuously differentiable functions.

Letting $L^*(q.\theta), f^*(q.\theta), j^*(q.\theta), J^*(q.\theta)$ and $\omega^*(q.\theta)$ to be respectively the transformed functions of $L(u.v)$, $\bar{f}(u.v)$, $\bar{j}(u.v)$, $\bar{J}(u.v)$ and $\bar{\omega}(u.v)$ in $(q.\theta)$ - coordinates, and using the above transformation relations in the equations of theorem 1, we obtain the following results:

Corollary. If $L^*(q.\theta)$ and $f^*(q.\theta)$ are the Legendre transform functions of a streamfunction and the proportionality function respectively of the equations governing the motion of steady plane orthogonal flow of an incompressible viscous fluid of finite electrical conductivity, then $L^*(q.\theta)$ and $f^*(q.\theta)$ must satisfy equations:

$$\zeta q\left[\cos\theta W_2^* + \sin\theta W_1^*\right] - \frac{\mu}{q}\left[\frac{\partial\left(\sin\theta\frac{\partial L^*}{\partial q} + \frac{\cos\theta}{q}\frac{\partial L^*}{\partial\theta}.\ J^*W_1^*\right)}{\partial(q.\theta)} + \frac{\partial\left(\cos\theta\frac{\partial L^*}{\partial q} - \frac{\sin\theta}{q}\frac{\partial L^*}{\partial\theta}.J^*W_2^*\right)}{\partial(q.\theta)}\right]$$
$$+\mu^{*2}\sigma f^*\left[\sin\theta\frac{\partial\left(\cos\theta\frac{\partial L^*}{\partial q} - \frac{\sin\theta}{q}\frac{\partial L^*}{\partial\theta}.Q^2F^*\right)}{\partial(q.\theta)} - \cos\theta\frac{\partial\left(\sin\theta\frac{\partial L^*}{\partial q} + \frac{\cos\theta}{q}\frac{\partial L*}{\partial\theta}.q^2f^*\right)}{\partial(q.\theta)}\right] = 0 \quad (3.26)$$

$$f^*\omega^* - J^*\left[\cos\theta\frac{\partial\left(\sin\theta\frac{\partial L^*}{\partial q} + \frac{\cos\theta}{q}\frac{\partial L^*}{\partial\theta}.f^*\right)}{\partial(q.\theta)} - \sin\theta\frac{\partial\left(\cos\theta\frac{\partial L^*}{\partial q} - \frac{\sin\theta}{q}\frac{\partial L^*}{\partial\theta}.f^*\right)}{\partial(q.\theta)}\right] = 0 \quad (3.27)$$

$$\mu^*\sigma f^*q^2 - J^*\left[\cos\theta\frac{\partial\left(\cos\theta\frac{\partial L^*}{\partial q} - \frac{\sin\theta}{q}\frac{\partial L^*}{\partial\theta}.f^*\right)}{\partial(q.\theta)} + \sin\theta\frac{\partial\left(\sin\theta\frac{\partial L^*}{\partial q} + \frac{\cos\theta}{q}\frac{\partial L^*}{\partial\theta}.f^*\right)}{\partial(q.\theta)}\right] = \mu^*\sigma K \quad (3.28)$$

where

$$J^*(q.\theta) = q^4\left[q^2\frac{\partial^2 L}{\partial q^2}\left\{q\frac{\partial L^*}{\partial q} + \frac{\partial^2 L^*}{\partial\theta^2}\right\} - \left\{\frac{\partial L^*}{\partial\theta} - q\frac{\partial^2 L^*}{\partial q\partial\theta}\right\}^2\right]^{-1} \quad (3.29)$$

$$\omega^*(q.\theta) = J^*\left[\frac{\partial^2 L^*}{\partial q^2} + \frac{1}{q^2}\frac{\partial^2 L^*}{\partial\theta^2} + \frac{1}{q}\frac{\partial L^*}{\partial q}\right] \quad (3.30)$$

$$W_1^*(q.\theta) = \frac{1}{q}\left[\frac{\partial\left(\sin\theta\frac{\partial L^*}{\partial q} + \frac{\cos\theta}{q}\frac{\partial L^*}{\partial\theta}.\omega^*\right)}{\partial(q.\theta)}\right] \quad (3.31)$$

$$W_2^*(q.\theta) = \frac{1}{q}\left[\frac{\partial\left(\cos\theta\frac{\partial L^*}{\partial q} - \frac{\sin\theta}{q}\frac{\partial L^*}{\partial\theta}.\omega^*\right)}{\partial(q.\theta)}\right] \quad (3.32)$$

$$j^* = \mu^*\sigma\left(f^*q^2 - K\right) \quad (3.33)$$

Once a solution $\{L^*(q.\theta).f^*(q.\theta)\}$ is known. we employ the relations

$$x = \sin\theta\frac{\partial L^*}{\partial q} + \frac{\cos\theta}{q}\frac{\partial L^*}{\partial\theta} \quad (3.34)$$

$$y = \frac{\sin\theta}{q}\frac{\partial L^*}{\partial\theta} - \cos\theta\frac{\partial L^*}{\partial q} \quad (3.35)$$

to obtain the velocity components in the physical plane. Having obtained $u = u(x.y).v = v(x.y)$. we get $f(x.y)$ in the $(x.y)$ - plane from $f^*(q.\theta)$. The other flow variables are then determined by using the flow equations in the physical plane.

4. Applications. In this section we study various flow problems as applications of theorem 1 and its corollary.

Application 1. (Vortex flow) Let

$$L^*(q,\theta) = F(q).F'(q) \neq 0.F''(q) \neq 0 \tag{4.1}$$

be the Legendre transform function. Using (4.1) to equations (3.29) to (3.32), we get

$$J^* = \frac{q}{F'(q)F''(q)}.\omega^*(q) = \frac{qF''(q) + F'(q)}{F'(q)F''(q)} \tag{4.2}$$

$$W_1^* = -\frac{1}{q}\omega^{*'}(q)\cos\theta F'(q).W_2^* = \frac{1}{q}\omega^{*'}(q)\sin\theta F'(q) \tag{4.3}$$

where prime denotes differentiation with respect to q.

Employing (4.2) and (4.3) in (3.26), (3.27) and (3.28), we obtain respectively

$$\frac{\mu}{q}\left\{\sin\theta\left[F''(q)(J^*W_1^*)_\theta + F'(q)(J^*W_2^*)_q\right]\right.$$
$$\left.+\cos\theta\left[F''(q)(J^*W_2^*)_\theta - F'(q)(J^*W_1^*)_q\right]\right\}$$
$$-\mu^{*2}\sigma f^* F'(q)\frac{\partial(q^2 f^*)}{\partial q} = 0 \tag{4.4}$$

$$\frac{\partial f^*}{\partial q} + \left[\frac{F''(q)}{F'(q)} + \frac{1}{q}\right] f^* = 0 \tag{4.5}$$

$$\frac{\partial f^*}{\partial \theta} - \mu^*\sigma q F'(q) f^* = -\mu^*\sigma K\frac{F'(q)}{q} \tag{4.6}$$

A general solution for (4.5) is

$$f^*(q,\theta) = \frac{\phi(\theta)}{qF'(q)} \tag{4.7}$$

where ϕ (θ) is an arbitrary function of θ.

Using (4.7) in (4.6), we get

$$\frac{1}{F'(q^2)}\frac{\phi'(\theta)}{\phi(\theta)} + \frac{\mu^*\sigma K}{\phi(\theta)} = \mu^*\sigma\frac{q}{F'(q)} \tag{4.8}$$

Differentiating (4.8) with respect to θ, we obtain

$$\phi''(\theta)\phi(\theta) - \phi'^2(\theta) - \mu^*\sigma K F'^2(q)\phi'(\theta) = 0 \tag{4.9}$$

We consider two cases: $\phi'(\theta) = 0$ and $\phi\ '\ (\theta\ ) \neq 0$.

(i) $\underline{\phi}'(\theta) \neq 0$

Equation (66) can be written as

$$\frac{\phi''(\theta)\phi(\theta) - \phi'^2(\theta)}{\phi'(\theta)} = \mu^*\sigma K F'^2(q) \tag{4.10}$$

Equation (4.10) implies that

$$\frac{\phi''(\theta)\phi(\theta) - \phi'^2(\theta)}{\phi'(\theta)} = C \tag{4.11}$$

$$\mu^*\sigma K F'^2(q) = C \tag{4.12}$$

where C is an arbitrary constant.

Since $F''(q) \neq 0$, it follows from equation (4.12) that

$$K = 0 \quad and \quad C = 0 \tag{4.13}$$

Hence, equation (4.11) reduces to

$$\frac{\phi''(\theta)}{\phi'(\theta)} - \frac{\phi'(\theta)}{\phi(\theta)} = 0 \tag{4.14}$$

Integrating (4.14) twice with respect to θ, we get

$$\phi(\theta) = D_2 exp[D_1\theta] \tag{4.15}$$

where D_1 and D_2 are nonzero arbitrary constants.
Using (4.13) and (4.15) in (4.8), we obtain

$$F'(q) = \frac{D_1}{\mu^*\sigma}\frac{1}{q} \tag{4.16}$$

Integrating (4.16) with respect to q, we have

$$F(q) = \frac{D_1}{\mu^*\sigma}\ell nq + D_3 \tag{4.17}$$

where D_3 is an arbitrary constant.
Substituting (4.15) and (4.16) in (4.7), we obtain

$$f^*(\theta) = \frac{\mu^*\sigma D_2}{D_1} \quad \exp[D_1\theta] \tag{4.18}$$

Employing (4.18) in the expression for $\omega^*(q)$ given in (4.2), we get

$$\omega^*(q) = 0 \tag{4.19}$$

Using (4.16), (4.18) and (4.19) in the equation (4.4), we obtain

$$\frac{2\mu^{*3}\sigma^2 D_2^2}{D_1^2}\exp(2D_1\theta) = 0$$

which is impossible since the left hand side of the above equation is greater than zero.
Therefore, $L^*(q) = \frac{D_1}{\mu^*\sigma}lnq + D_3$ is not the Legendre transform function of the stream-function of the flow.

(ii) $\underline{\phi}(\theta) = 0$

In this case, we have

$$\phi(\theta) = D_4 = constant \tag{4.20}$$

Employing (4.20) in (4.8), we get

$$F'(q) = \frac{D_4}{K}q, \quad K \neq 0 \tag{4.21}$$

Hence,

$$F(q) = \frac{D_4}{2K} \quad q^2 + D_5 \tag{4.22}$$

From (4.2), (4.7), (4.20) and (4.22), we obtain

$$f^*(q) = \frac{K}{q^2} \tag{4.23}$$

$$\omega^* = \frac{2K}{D_4} \tag{4.24}$$

Using (4.21). (4.23) and (4.24) in equation (4.4). we find that equation (4.4) is identically satisfied. Therefore.

$$L^*(q.\theta) = F(q) = \frac{D_4}{2K}q^2 + D_5 . K \neq 0 \tag{4.25}$$

is a Legendre transform function of the flow for which $f^*(q)$ is given by (4.23).

Writing (4.25) in the $(u.v)$ - plane. we have

$$L(u.v) = \frac{D_4}{2K}(u^2 + v^2) + D_5 \tag{4.26}$$

Employing (4.26) in (3.13). we obtain

$$\mathbf{V}(x.y) = \frac{K}{D_4}(-y.x) \tag{4.27}$$

From (4.23) and (4.27). we determine

$$f(x.y) = \frac{D_4^2}{K(x^2 + y^2)}$$

$$\mathbf{H}(x.y) = f(x.y)(-v.u) = \frac{-D_4}{x^2 + y^2}(x.y) \tag{4.28}$$

Using (4.24). (4.27). (4.28) in equations (2.9) and (2.10). and integrating. we obtain

$$h(x.y) = \frac{\zeta K^2}{D_4^2}(x^2 + y^2) + D_6 \tag{4.29}$$

where D_6 is an arbitrary constant.

From (2.7). we determine the pressure to be

$$p(x.y) = \frac{\zeta K^2}{2D_4^2}(x^2 + y^2) + D_6 \tag{4.30}$$

Summing up, we have the following theorem:

Theorem 2. If $L^*(q.\theta\ ) = F(q)$ is the Legendre transform function of streamfunction for a steady. plane. orthogonal flow of an incompressible viscous fluid of finite electrical conductivity. then the flow in the physical plane is a vortex flow given by equations (4.27), (4.28) to (4.30).

Application 2. (Radial flow)

We let

$$L^*(q.\theta) = A\theta + B. A \neq 0 \tag{4.31}$$

to be the Legendre transform funciton. where A and B are arbitrary constants.

Evaluating J^*. ω *. W_1^* and W_2 * as before. we get

$$J^* = -\frac{q^4}{A^2}. \omega^* = W_1^* = W_2^* = 0 \tag{4.32}$$

Following the same analysis as in previous applications. we obtain the system of equations

$$\frac{\partial f^*}{\partial \theta} = 0 \tag{4.33}$$

$$\frac{\partial f^*}{\partial q} + \frac{\mu^* \sigma A}{q} f^* = \frac{\mu^* \sigma K A}{q^3} \tag{4.34}$$

Equations (4.33) yields

$$f^* = f^*(q) \tag{4.35}$$

Therefore, equation (4.34) becomes an ordinary differential equation which has the general solutions:

(i) $\underline{\mu^*\sigma A \neq 2}$

$$f^*(q) = \frac{\mu^*\sigma K A}{\mu^*\sigma A - 2} q^{-2} + E_1 q^{-\mu^*\sigma A} \tag{4.36}$$

(ii) $\underline{\mu^*\sigma A = 2}$

$$f^*(q) = \frac{2K}{q^2}\ell n q + \frac{E_2}{q^2} \tag{4.37}$$

where E_1 and E_2 are arbitrary constants.

Proceeding as before, we obtain the following theorem:

Theorem 3. If $L^*(q, \theta) = A\theta + B$ is the Legendre transform function of streamfunction for a steady, plane, orthogonal flow of a viscous incompressible fluid of finite electrical conductivity, then the flow in the physical plane is a radial flow given by equations:

(i) $\underline{A \neq 2/\mu^*\sigma}$

$$\mathbf{V}(x, y) = \left(\frac{Ax}{x^2 + y^2}, \frac{Ay}{x^2 + y^2}\right)$$

$$\omega = 0$$

$$f(x, y) = \frac{\mu^*\sigma K A}{\mu^*\sigma A - 2} q^{-2} + E_1 q^{-\mu^*\sigma A}$$

where

$$q^2 = A^2/(x^2 + y^2)$$

$$\mathbf{H}(x, y) = \left[\frac{\mu^*\sigma K A}{\mu^*\sigma A - 2} q^{-2} + E_1 q^{-\mu^*\sigma A}\right]\left(\frac{-Ay}{x^2 + y^2}, \frac{Ax}{x^2 + y^2}\right) \tag{4.38}$$

$$j(x, y) = \mu^*\sigma E_1 q^{(2-\mu^*\sigma A)} + \frac{2\mu^*\sigma K}{\mu^*\sigma A - 2} \tag{4.39}$$

$$p(x, y) = \mu^* \int jH_2 dx + jH_1 dy - \frac{1}{2}\zeta A^2/(x^2 + y^2) \tag{4.40}$$

where j, H_1 and H_2 are given by (4.38) and (4.39).

(ii) $\underline{A = 2/\mu^*\sigma}$

$$\mathbf{V}(x, y) = \frac{2}{\mu^*\sigma}\left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right)$$

$$\omega = 0$$

$$f(x, y) = \frac{2K}{q^2}\ell n q + \frac{E_2}{q^2}$$

where

$$q^2 = \frac{4}{(\mu^*\sigma)^2(x^2 + y^2)}$$

$$\mathbf{H}(x, y) = \frac{2}{\mu^*\sigma}\left[\frac{2K}{q^2}\ell n q + \frac{E_2}{q^2}\right]\left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right) \tag{4.41}$$

$$j(x, y) = \mu^*\sigma K(\ln q^2 - 1) + \mu^*\sigma E_2 \tag{4.42}$$

and the pressure is given by (4.40) where now j, H_1 and H_2 are given by (4.41) and (4.42).

Application 3. (Spiral flow)

Letting

$$L^*(q, \theta) = G_1 \ell n q + G_2 \theta \tag{4.43}$$

to be the Legendre transform function, where G_1 and G_2 are non-zero arbitrary constants.

Evaluating J^*, ω^*, W_1^* and W_2^* as before, we get

$$J^* = \frac{-q^4}{G_1^2 + G_2^2} . \omega^* = W_2^* = W_1^* = 0 \tag{4.44}$$

Proceeding as in previous applications. we obtain the following system of equations:

$$\frac{2G_1}{q^2} f^* + \frac{G_1}{q}\frac{\partial f^*}{\partial q} + \frac{G_2}{q^2}\frac{\partial f^*}{\partial \theta} = 0 \tag{4.45}$$

$$\frac{G_1}{q}\frac{\partial f^*}{\partial q} + \frac{G_2}{q^2}\frac{\partial f^*}{\partial \theta} = 0 \tag{4.46}$$

$$\mu^* \sigma \quad q^2 f^* + \frac{q^2}{G_1^2 + G_2^2}\left[G_2 q \frac{\partial f^*}{\partial q} - G_1 \frac{\partial f^*}{\partial \theta}\right] = \mu^* \sigma K \tag{4.47}$$

From (4.45) and (4.46). we get

$$f^* = 0$$

which is a trivial solution for the system of equations (4.45) to (4.47). Therefore. the spiral flow is not possible.

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