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SOME REMARKS ON INEQUALITIES THAT CHARACTERIZE INNER PRODUCT SPACES

JAVIER ALONSO

Departamento de Matemáticas
Universidad de Extremadura
06071 Badajoz, Spain

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ABSTRACT. The purpose of this paper is to obtain some new characterizations of inner product spaces in the line of the Jordan–von Neumann identity and to make some remarks on similar known characterizations of Day, Delbosco, Rassias and Senechalle, among others.

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Let E be a real normed linear space. It is well known that E is an inner product space (i.p.s.) if and only if the "parallelogram equality" of P. Jordan and J. von Neumann [1]

$$\|x+y\|^2+\|x-y\|^2=2(\|x\|^2+\|y\|^2)$$

holds for every $x,y\in E$. The aim of this paper is to give an account of some characterizations of inner product spaces weaker than the one above in order to make some remarks about them and to obtain some new characterizations of the same type.

We shall use S to denote the unit sphere of E, and $H(x,y)$ the function

$$H(x,y)=\|x+y\|^2+\|x-y\|^2-2(\|x\|^2+\|y\|^2) \qquad (x,y\in E).$$

The first weakening of the Jordan–von Neumann condition was given by M.M. Day [2] by proving that it can be restricted to unit vectors (i.e. the "rhombus equality"):

$$E \text{ is an i.p.s.} \Leftrightarrow H(x,y)=0 \text{ for every } x,y\in S.$$

Then I.J. Schoenberg [3] proved that the rhombus equality can be replaced by the "rhombus inequality":

$$E \text{ is an i.p.s.} \Leftrightarrow H(x,y)\sim 0 \text{ for every } x,y\in S,$$

where $\sim$ denotes either $\geq$ or $\leq$.

The parallelogram equality can also be changed into the "rectangle inequality" of C. Benítez and M. del Río [4]:

$$E \text{ is an i.p.s.} \Leftrightarrow H(x,y)\sim 0 \text{ for every } x,y\in E,\ x\perp y,$$

where $x\perp y$ means "orthogonality" in the G. Birkhoff [5] sense (i.e. $\|x+\lambda y\|\geq\|x\|$ for every $\lambda\in\mathbb{R}$).

As D. Amir pointed out [6], implicit in the proof of the above result is that it remains valid if Birkhoff–orthogonality is replaced by any other relation R in E that likewise "admits diagonals" (i.e. such that for every $x,y\in E\setminus\{0\}$ there exists $\alpha>0$ such that $x+\alpha y\,R\,x-\alpha y$) [7]. This happens, for example, if $\perp$ denotes any of the orthogonalities Carlsson [8,9], Diminnie [10,11] and Area [12,13]. (Recall that an orthogonality in a normed linear space is a relation that coincides with the usual orthogonality when the norm is induced by an inner product.)

It is known [5,2,14,15] that if $\dim E\geq 3$ the symmetry of Birkhoff–orthogonality is an inner product space characteristic, and that there exist two–dimensional non inner product spaces (characterized by M.M. Day [2]) with this property satisfied. Nevertheless, it was proved also in [4] that if E is a two–dimensional normed linear space with symmetric B–orthogonality that satisfies a "square inequality" then E is an inner product space. This can be expressed in the following way:

$$E \text{ is an i.p.s.} \iff [x,y\in S,\ x\perp^B y \implies y\perp^B x \text{ and } H(x,y)\leq 0].$$

However, there exist non–inner product spaces with symmetric B–orthogonality and $H(x,y)\geq 0$ for every $x,y\in S$, $x\perp^B y$ (i.e. $\mathbb{R}^2$ endowed with a norm whose spheres are regular hexagons [4]). It is an open question whether the square inequality ($\leq$) alone is characteristic of inner product spaces.

As a particular case of a more general result, D.A. Senechalle [16] proved that if $\epsilon>0$ then

$$E \text{ is an i.p.s.} \iff [x,y\in S,\ \|x-y\|<\epsilon \implies H(x,y)\sim 0].$$

Pursuing this line further, and taking into account that the relation "$x\,R\,y$ iff $\|x-y\|=\epsilon$" admits diagonals, we can say that if $\epsilon>0$ then

$$E \text{ is an i.p.s.} \iff [x,y\in E,\ \|x-y\|=\epsilon \implies H(x,y)\sim 0].$$

In the next proposition we shall see that, for particular values of ϵ, these preceding two characterizations can be improved.

PROPOSITION. Let $D=\{2\cos\frac{k\pi}{2n}\colon n=2,3,\ldots,\ k=1,2,\ldots,n-1\}\subset[0,2]$. If $\epsilon\in(0,2)\setminus D$ then

$$E \text{ is an i.p.s.} \iff [x,y\in S,\ \|x-y\|=\epsilon \implies H(x,y)\sim 0],$$

where $\sim$ denotes either $\geq$ or $\leq$.

PROOF. It is proved in [17] that if $\epsilon\in(0,2)\setminus D$ then only the inner product spaces satisfy the property

$$Q_\epsilon\colon \qquad x,y\in S,\ \|x-y\|=\epsilon \implies H(x,y)=0.$$

On the other hand, G. Nordlander [18] proved that if $S_{\mathbb{R}^2}$ is the unit sphere of a norm in $\mathbb{R}^2$ and $0\leq\epsilon\leq 2$ then the set $\{x+y\colon x,y\in S_{\mathbb{R}^2},\ \|x-y\|=\epsilon\}$ is a simple closed curve that bounds an area $(4-\epsilon^2)$–times the area bounded by $S_{\mathbb{R}^2}$. Thus, any normed linear space that satisfies the property

$$x,y\in S,\ \|x-y\|=\epsilon \implies H(x,y)\sim 0$$

must also satisfy Q_ϵ.

However, for $\epsilon\in D$ there exist non–inner product spaces that satisfy Q_ϵ (i.e. if E is the vector space $\mathbb{R}^2$ endowed with a norm whose spheres are 4n–gons then it satisfies Q_ϵ for $\epsilon=2\cos\frac{k\pi}{2n}$, $k=1,2,\ldots,n-1$) [17].

In an extensive paper J. Oman [19] proved that, for two–dimensional normed linear spaces, one of the vectors in the parallelogram inequality ($\leq$) can be fixed. That is, if $\dim E=2$ then

$$E \text{ is an i.p.s.} \iff \text{there exists an } x\in E \text{ such that } H(x,y)\leq 0 \text{ for every } y\in E.$$

However, this result is not true for ($\geq$). If E is $\mathbb{R}^2$ endowed with the norm $\|(x_1,x_2)\|=|x_1|+|x_2|$ and $x=(1,0)$ then $H(x,y)\geq 0$ for every $y\in E$ [19].

It is essential for the Oman characterization that dimE=2. If, for example, we take E to be the vector space $\mathbb{R}^3$ endowed with the norm whose unit ball is the set of points (x_1,x_2,x_3) defined by the intersection of the two circular cylinders $x_1^2+x_2^2\leq 1$, $x_2^2+x_3^2\leq 1$, then any plane containing the origin and the point $x=(0,0,1)$ intersects the unit sphere in an ellipse. Therefore, $H(x,y)=0$ for every $y\in E$. (Similar results for dimensions greater than two can be found in [19].)

It is an open question whether one vector can also be fixed in the rhombus inequality ($\leq$).

With regard to norm inequalities of higher degree, Yang Cong–Ren [20] proved that if $p>2$ then

$$E \text{ is an i.p.s.} \iff \|x+y\|^p+\|x-y\|^p \geq 2(\|x\|^p+\|y\|^p) \text{ for every } x,y\in E.$$

(This inequality does not hold for $p<2$. Similar inequalities for this case have been obtained in [20].)

D. Delbosco [21] proved that if there exists $p>0$ such that $\|x+y\|^p + \|x-y\|^p=2^p$ for every $x,y\in S$, then E must be an inner product space. Therefore, taking into account that if $x,y\in S$ are orthogonal then $\|x\pm y\|=2^{1/2}$, it follows that $p=2$. Moreover, taking $\epsilon=2^{1/2}$ in the Nordlander sets defined above, it follows that in any two–dimensional subspace of any normed linear space there always exist $x,y\in S$ such that $\|x+y\|=\|x-y\|=2^{1/2}$. Therefore, Delbosco's identity holds only for $p=2$.

Finally, we shall consider the T.M. Rassias [22] characterization:

$$E \text{ is an i.p.s.} \iff \|x+y\|^p+\|x-y\|^p\leq 2^{p-1}(\|x\|^p+\|y\|^p) \text{ for every } x,y\in E \text{ and } p>2.$$

Rassias's proof of this result is based on making p tend to 2 in order to obtain the rhombus inequality. Some observations are suggested by this characterization:

Firstly, from the arguments given above, Rassias's inequality does not hold if we change "$\leq$" into "$\geq$".

Secondly, we cannot fix p. For any $p\geq 2$ the spaces ℓ^p and L^p both satisfy the inequality

$$\|x+y\|_p^p+\|x-y\|_p^p \leq 2^{p-1}(\|x\|_p^p+\|y\|_p^p)$$

(see e.g. Köthe [23], p.356). Furthermore, we shall see that the Rassias characterization does not hold if the inequality holds for every $p\geq r$ for some $r>2$. This proves that the hypothesis on p in the Rassias characterization of inner product spaces cannot be weakened.

Taking into account that, for every $c\geq 0$, the functions $f(x)=(1+c^x)^{1/x}$ and $g(x)=\left[\frac{1+c^x}{2}\right]^{1/x}$, $x\geq 0$, are, respectively, monotone decreasing and monotone increasing, it is easy to see that for every $a\geq 0$, $b\geq 0$, $0\leq r\leq p$, the following two inequalities hold:

$$(a^p+b^p)^{1/p} \leq (a^r+b^r)^{1/r}$$

$$\left[\frac{a^p+b^p}{2}\right]^{1/p} \geq \left[\frac{a^r+b^r}{2}\right]^{1/r}.$$

Bearing this in mind, we have therefore proved that for $p\geq r>2$ and every x,y in ℓ^r or L^r, according to the case,

$$(\|x+y\|_r^p+\|x-y\|_r^p)^{1/p} \leq (\|x+y\|_r^r+\|x-y\|_r^r)^{1/r} \leq$$
$$\leq 2^{1-1/r}(\|x\|_r^r+\|y\|_r^r)^{1/r} \leq 2^{1-1/p}(\|x\|_r^p+\|y\|_r^p)^{1/p}.$$

Hence

$$\|x+y\|_r^p+\|x-y\|_r^p \leq 2^{p-1}(\|x\|_r^p+\|y\|_r^p).$$

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