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A NOTE ON A FUNCTIONAL INEQUALITY

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ABSTRACT. We prove: If $r_1, \ldots, r_k$ are (fixed) positive real numbers with $\prod_{j=1}^{k} r_j > 1$, then the only entire solutions $\varphi\colon \mathbb{C} \to \mathbb{C}$ of the functional inequality

$$\prod_{j=1}^{k} |\varphi(r_j z)| \ge \left(\prod_{j=1}^{k} r_j\right) |\varphi(z)|^k$$

are $\varphi(z) = cz^n$, where c is a complex number and n i s a positive integer.

KEY WORDS AND PHRASES. Functional inequality, entire functions.
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1. INTRODUCTION.

Inspired by a problem of H. Haruki, who asked for all entire solutions of

$$|\varphi(z+w)|^2 + |\varphi(z-w)|^2 + 2|\varphi(0)|^2 \ge 2|\varphi(z)|^2 + 2|\varphi(w)|^2, \tag{1.1}$$

J. Walorski [1] proved in 1987 the following interesting proposition:

Let $r > 1$ be a (fixed) real number. Then the only entire solutions $\varphi\colon \mathbb{C} \to \mathbb{C}$ of the functional inequality

$$|\varphi(rz)| \ge r|\varphi(z)|$$

are

$$\varphi(z) = cz^n, \tag{1.2}$$

where $c \in \mathbb{C}$ and $n \in \mathbb{N}$.

As an application of this theorem, Walorski showed that the only entire functions $\varphi\colon \mathbb{C} \to \mathbb{C}$ satisfying (1.1) and $\varphi(0) = 0$ are the monomials (1.2). The aim of this note is to prove an extension of Walorski's result by using a method which is (slightly) different from the two approaches presented in [1].

2. MAIN RESULTS.

Theorem. Let $r_1, \ldots, r_k$ be (fixed) positive real numbers with $\prod_{j=1}^{k} r_j > 1$. Then the only entire solutions $\varphi\colon \mathbf{C} \to \mathbf{C}$ of

$$\prod_{j=1}^{k} |\varphi(r_j z)| \geq (\prod_{j=1}^{k} r_j) |\varphi(z)|^k \tag{2.1}$$

are the functions $\varphi(z) + cz^n$, where c is a complex number and n is a positive integer.

PROOF. Simple calculations reveal that the functions $\varphi(z) = cz^n$ ($c \in \mathbf{C}$, $n \in \mathbf{N}$) satisfy (2.1). Next we assume that φ is an entire solution of inequality (2.1).

Because of $\prod_{j=1}^{k} r_j > 1$ we conclude from (2.1) with $z = 0$ that φ has at 0 a zero. Let n be the order of this zero; we define

$$f(z) = \varphi(z)/z^n\,, \tag{2.2}$$

then f is an entire function with $f(0) \neq 0$. From (2.1) we obtain

$$\prod_{j=1}^{k} |f(r_j z)| \geq (\prod_{j=1}^{k} r_j^{1-n}) |f(z)|^k\,. \tag{2.3}$$

We suppose that f has a zero z_0. By induction it follows from (2.3) that z_0/r_1^m is a root of f for all non-negative integers m. From the identity theorem we conclude $f(z) \equiv 0$ which contradicts the condition $f(0) \neq 0$. Hence f has no zero which implies that the function

$$g(z) = \frac{f(z)^k}{\prod_{j=1}^{k} f(r_j z)} \tag{2.4}$$

is entire. From (2.3) we conclude

$$|g(z)| \leq \prod_{j=1}^{k} r_j^{n-1} \qquad \text{for all } z \in \mathbf{C}\,,$$

and Liouville's theorem implies that g is a constant. Therefore we have

$$f(z)^k = K \prod_{j=1}^{k} f(r_j z), \qquad K \in \mathbf{C}\,. \tag{2.5}$$

Since $f(0) \neq 0$ we get from (2.5): $K = 1$;
hence

$$f(z)^k = \prod_{j=1}^{k} f(r_j z)\,. \tag{2.6}$$

Differentiation leads to

$$k\,\frac{f'(z)}{f(z)} = \sum_{j=1}^{k} r_j \frac{f'(r_j z)}{f(r_j z)}\,. \tag{2.7}$$

Setting

$$\frac{f'(z)}{f(z)} = \sum_{m=0}^{\infty} a_m z^m \tag{2.8}$$

we obtain from (2.7) and (2.8):

$$\sum_{m=0}^{\infty} k a_m z^m = \sum_{m=0}^{\infty} (a_m \sum_{j=1}^{k} r_j^{m+1}) z^m\,, \tag{2.9}$$

and comparing the coefficients of z^m yields for all $m \geq 0$:

$$ka_m = a_m \sum_{j=1}^{k} r_j^{m+1} . \tag{2.10}$$

We assume that there exists an integer $m_0 \geq 0$ such that $a_{m_0} \neq 0$, then we get from the arithmetic mean-geometric mean inequality and from (2.10):

$$\left[\sum_{j=1}^{k} r_j^{m_0+1}\right]^{1/k} \leq \frac{1}{k} \sum_{j=1}^{k} r_j^{m_0+1} = 1 ,$$

which contradicts the assumption $\sum_{j=1}^{k} r_j > 1$. Hence, $a_m = 0$ for all $m > 0$. This implies that f is a constant, say $c \in \mathbb{C}$, and therefore we obtain $\varphi(z) = cz^n$.

It is natural to look for all entire functions $\varphi: \mathbb{C} \to \mathbb{C}$ which satisfy the following additive counterpart of inequality (2.1):

$$\left(\sum_{j=1}^{k} \varphi(r_j z)\right) \geq \sum_{j=1}^{k} r_j \; |\varphi(z)| \; , \tag{2.11}$$

where $r_1, \ldots r_k$ are (fixed) positive real numbers with $\sum_{j=1}^{k} r_j > k$. The monomials $\varphi(z) = cz^n (c \in \mathbb{C}, n \in \mathbb{N})$ are solutions of (2.11). Indeed, inequality (2.11) with $\varphi(z) = cz^n$ reduces to

$$\sum_{j=1}^{k} r_j^n \geq \sum_{j=1}^{k} r_j , \tag{2.12}$$

which follows immediately from Jensen's inequality and the assumption $\sum_{j=1}^{k} r_j > k$. By an argumentation similar to the one we have used to establish the theorem it can be shown that the functions $\varphi(z) cz^n$ $(c \in \mathbb{C}, n \in \mathbb{N})$ are the only entire solutions of (2.11). This provides another extension of Walorski's result.

If the expression on the left-hand side of (2.11) will be replaced by $\sum_{j=1}^{k} |\varphi(r_j z)|$, then we conclude from the triangle inequality that $\varphi(z) = cz^n$ $(c \in \mathbb{C}, n \in \mathbb{N})$ also solve

$$\sum_{j=1}^{k} |\varphi(r_j z) \geq \sum_{j=1}^{k} r_j |\varphi(z)| \; , \tag{2.13}$$

where $r_1, \ldots, r_k$ are (fixed) positive real numbers with $\sum_{j=1}^{k} r_j > k$. We finish by asking: Are there more solutions of (2.13) (if $k > 1$)?

REFERENCE

1. WALORSKI, J., On a functional inequality, Aequationes Math. 32 (1987), 213-215.