Internat. J. Math. & Math. Sci.
VOL. 15 NO. 2 (1992) 355-358

ON THE STRUCTURE OF SELF ADJOINT ALGEBRA OF FINITE STRICT MULTIPLICITY

S.C. ARORA and PAWAN BALA

Department of Mathematics
University of Delhi
Delhi-110 007, INDIA

(Received February 2, 1989, and in revised form April 30, 1989)

KEY WORDS AND PHRASES. Algebras of finite strict multiplicity, projections, self-adjoint, weakly closed algebra.

1980 AMS SUBJECT CLASSIFICATION CODES. Revision Primary 47C05, Secondary 47D25.

1. INTRODUCTION.

Throughout this paper X is a complex Hilbert space. For any subset M of X, $B(M)$ denotes the algebra of all bounded linear operators on M. An algebra $\mathcal{A}$ means a strongly closed subalgebra of $B(X)$ containing the identity element I. $\mathcal{A}$ is said to be an algebra of finite strict multiplicity (a.f.s.m.), if there exists a finite subset $\Gamma = \{x_1, x_2, \dots, x_n\}$ of X such that $\mathcal{A}(\Gamma) = \{A_1x_1 + A_2x_2 + \dots + A_nx_n,\ A_i \in \mathcal{A}\} = X$. In this case we denote the algebra by $(\mathcal{A}, \{x_i\}_{i=1}^n)$. If $n = 1$, i.e., if there exists a vector x_0 such that $\mathcal{A}x_0 = \{Ax_0 : A \in \mathcal{A}\} = X$, then $\mathcal{A}$ is said to be a strictly cyclic algebra. In this case vector x_0 is called a strictly cyclic vector for $\mathcal{A}$. Algebra $\mathcal{A}$ is said to be self-adjoint, if $A^* \in \mathcal{A}$, whenever A is in $\mathcal{A}$. For any subset $\mathfrak{B}$ of $B(X)$, the commutant of $\mathfrak{B}$, denoted by $\mathfrak{B}'$, is the collection of all operators in $B(X)$ that commute with $\mathfrak{B}$.

A closed linear subspace M of X reduces the subset $\mathfrak{B}$ of $B(X)$, if the projection of X onto M is in $\mathfrak{B}'$. A collection $\{M_j\}$ of closed linear subspaces of X is said to be an orthogonal decomposition of X, if the Mj 's are pair-wise orthogonal and span X. Correspondingly, a collection $\{P_j\}$ of projections, is said to be a resolution of identity, if the collection $\{P_j(X)\}$ of ranges of P_i forms an orthogonal decomposition of X.

Strictly cyclic operator algebras have been studied by Lambert [1], [2], M. Embry [3], [4], [5], Bolstein [6] and others. The study of strictly cyclic algebras was extended to that of algebras of finite strict multiplicity by Herrero in [7], [8]. This paper aims at studying the structure of the commutant of an a.f.s.m., particularly a self-adjoint a.f.s.m. in terms of its reducing subspaces. By [5], the commutant of a self-adjoint strictly cyclic algebra cannot have any infinite collection of pair-wise orthogonal projections. [9, Theorem 2] paves the way for the following:

2. MAIN RESULTS.

THEOREM 1. If $(\mathcal{A}, \{x_i\}_{i=1}^n)$ is an a.f.s.m. on X, then each collection of mutually-orthogonal projections in $\mathcal{A}'$ is finite.

PROOF. Let $\{P_j\}$ be a collection of mutually-orthogonal projections in $\mathcal{A}'$. We may assume $\{P_j\}$ to be countable. Let $Q_n = \sum_{j=1}^{n} P_j$ and $Q = \sum_{j=1}^{\infty} P_j$. Q_n converges strongly to Q. By [9, Theorem 2], Q_n converges uniformly to Q. As $Q - Q_n$ is a projection, its norm is zero or one. Since $\| Q_n - Q \|$ can be made arbitrarily small, there exists m such that $\| Q_n - Q \| = 0$ for all $n \geq m+1$. This implies that the collection $\{P_j\} = \{P_j\}_{j=1}^{m}$ is finite.

COROLLARY 2. Let $(\mathcal{A}, \{x_i\}_{i=1}^{n}$ be an a.f.s.m. on X. Any operator in $\mathcal{A}'$ with residual spectrum empty is of finite spectrum.

PROOF. Let E in $\mathcal{A}'$ have residual spectrum empty. By [8], E has no continuous spectrum. Therefore, spectrum of E consists entirely of point spectrum. By Theorem 1, E has only finite number of distinct eigenspaces. So spectrum of E is finite.

Our next theorem generalizes [5, Theorem 3] to a self-adjoint a.f.s.m.

THEOREM 3. Let $\mathcal{A}$ be a self-adjoint a.f.s.m. on X. Then there exists a finite orthogonal decomposition $\{M_k\}$ of X such that each M_k reduces $\mathcal{A}$, and $\mathcal{A} \mid_{M_k}$ is strongly dense in $B(M_k)$.

PROOF. If X and $\{0\}$ are the only reducing subspaces of $\mathcal{A}$, then by [8], $\mathcal{A}$ is strongly dense in $B(X)$. As such the trivial decomposition $\{X\}$ of X satisfies the requirements of the theorem.

Let $\{M_k\}_{k=1}^{p}$ be a collection of mutually orthogonal subspaces of X such that each M_k reduces $\mathcal{A}$, and $\mathcal{A} \mid_{M_k}$ is strongly dense in $B(M_k)$. If these M_k's span X, the theorem follows. Otherwise,

consider $A_1 = \mathcal{A} \mid [\bigvee_{i=1}^{P} M_i]^{\perp}$. Let P be orthogonal projection of X onto $M = [\bigvee_{i=1}^{P} M_i]^{\perp}$. P is in $\mathcal{A}'$ and $(\mathcal{A}, \{Px_i\}_{i=1}^{n})$ is an a.f.s.m. on M. If $\mathcal{A}_1$ has only trivial reducing subspace, then again, by [8], $\mathcal{A}_1$ is strongly dense in $B(M)$ and the construction is complete. Otherwise, $\mathcal{A}_1$ has a non-trivial reducing subspace. This implies $\mathcal{A}_1$ has a minimal reducing subspace, say M_{p+1}. By [8], $\mathcal{A}_1 \mid_{Mp+1}$ is strongly dense in $B(M_{p+1})$. Thus, M_1, $M_2, \ldots, M_{p+1}$ are pair-wise orthogonal subspaces for $\mathcal{A}$, and $\mathcal{A} \mid_{M_k}$ is strongly dense in $B(M_k)$ for $k = 1, 2, \ldots, p+1$. By Theorem 1, the collection terminates with a finite number of pair-wise orthogonal reducing subspaces.

Our next theorem depicts the structure of commutant of a self-adjoint a.f.s.m. The theorem and its consequences can be proved by following the technique used by Embry in [5]. So we omit the proofs.

THEOREM 4. Let $(\mathcal{A}, \{x_i\}_{i=1}^{n})$ be a self-adjoint a.f.s.m., $\{M_k\}$ a decomposition of X as required in Theorem 3 and P_k the orthogonal projection of X onto M_k. Then $\mathcal{A}' = \sum P_j \mathcal{A}' P_k$ and $P_j \mathcal{A}' P_k$ is of dimension one or zero for each value of j and k. In particular, $\mathcal{A}'$ is finite-dimensional.

COROLLARY 5. If $\mathcal{A}$ is self-adjoint a.f.s.m. with an abelian commutant, then $\mathcal{A}' = \{\sum_{j=1}^{n} \lambda_j P_j : \lambda_j$ is complex$\}$, wherein $\{P_j\}$ are projections as required in Theorem 4. In particular, $\{P_j\}$ consists of normal operators with finite spectra.

COROLLARY 6. Let N be a normal operator with $\{N\}'$ as an a.f.s.m. Then there exist orthogonal projections P_1, $P_2, \ldots, P_n$ such that $\{N\}'' = \{\sum_{j=1}^{n} \lambda_j P_j$, λ_j complex$\}$.

COROLLARY 7. The decomposition $\{M_k\}$ in Theorem 4 is unique, if and only if, $\mathcal{A}'$ is abelian.

If $\mathcal{A}$ is any a.f.s.m. on X, then $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, where $\mathcal{A}_1$ is a self-adjoint a.f.s.m., and $\mathcal{A}_2$ is an a.f.s.m. having no reducing subspaces on which it is self-adjoint.

An operator T in $B(X)$ is said to be of finite strict multiplicity, if the weakly closed algebra $\mathcal{A}(T)$ generated by T and I is of finite strict multiplicity. Our next theorem extends [10, Theorem 6] proved by Barnes.

THEOREM 8. For an operator T in $B(X)$, let $\{x_1, x_2, ..., x_n\}$ be a subset of X such that $(\mathcal{A}(T), \{x_i\}_{i=1}^n)$ is an a.f.s.m. on X. Then there exists a finite mutually-orthogonal collection of subspaces $\{X_1, X_2, ..., X_n\}$ of X satisfying the following

(i) Each X_i reduces T

(ii) $X = X_1 \oplus X_2 \oplus ... \oplus X_k$ and thus $T = T_1 \oplus T_2 ... \oplus T_k$, where $T_i = T\mid_{X_i}$.

(iii) Each T_j is an irreducible operator of finite strict multiplicity on X_j.

PROOF. Let $\mathfrak{B}$ be a closed subalgebra of $B(X)$ generated by T, T^* and I. Define a positive functional f on $\mathfrak{B}$ by $f: \mathfrak{B} \to C$ as $f(S) = (Sx_o, x_o)$, where $x_o = x_1 + x_2 + ... + x_n$, S in $\mathfrak{B}$. Let $K_f = \{S \in \mathfrak{B};\ f(S^*S) = 0\} = \{S \in \mathfrak{B}: Sx_o = 0\}$. There are two norms on the quotient space $\mathfrak{B}/K_f$, viz.

$$\text{(i)}\quad \| A + K_f \|_f = f(A^*A)^{1/2} = \| Ax_o \|$$

$$\text{(ii)}\quad \| A + K_f \| = \inf \{ \| A - K \| : K \in K_f \}.$$

These norms are related by $\| A + K_f \|_f \leq \| x_o \| \, \| A + K_f \|$ for all A in $\mathfrak{B}$. $\mathfrak{B}/K_f$ is complete w.r.t. both these norms. By closed graph theorem, the norms are equivalent on $\mathfrak{B}/K_f$. By Halpern [11], the commutant $\mathfrak{B}'$ of $\mathfrak{B}$ in $B(X)$ has the following properties:

(i) If F is a non-zero projection in $\mathfrak{B}'$, then F majorizes a minimal projection E in $\mathfrak{B}'$;

(ii) A maximal set of mutually orthogonal projections in $\mathfrak{B}$ must be finite.

By (i), we can choose a non-empty maximal set of mutually orthogonal projections in $\mathfrak{B}'$ and, by (ii), this set is finite. Let $\{E_1, E_2, ..., E_k\}$ be this set. Let $X_j = R(E_j)$, $j = 1, 2, ..., k$. Then $X = X_1 \oplus X_2 \oplus ... \oplus X_k$. The collection $\{X_1, X_2, ..., X_k\}$ reduces $\mathfrak{B}$; and $\mathfrak{B}$ acts irreducibly on each X_j, $j = 1, 2, ..., k$. Now $x_i \in X$ implies $x_i = x_{il} \oplus x_{i2} \oplus ... \oplus x_{ik}$, $\iota = 1, 2, ..., n$ where $x_{ij} \in X_j$ for all $j = 1, 2, ..., k$. For $y \in X_j \subset X$, there exist operators $R_1, R_2, ... R_n$ in $A(T)$ such that

$$y = R_1x_1 + R_2x_2 + ... + R_nx_n$$

$$= R_1(x_{11} \oplus x_{12} \oplus ... \oplus x_{lk}) + R_2(x_{21} \oplus ... \oplus x_{2k}) + ... + R_n(x_{n1} \oplus x_{n2} \oplus ... \oplus x_{nk})$$

$$= (R_1x_{11} + R_2x_{21} + ... + R_nx_{n1}) \oplus (R_1x_{12} + R_2x_{22} + ... + R_nx_{n2}) \oplus ... \oplus$$

$$\oplus (R_1x_{1k} + R_2x_{2k} + ... + R_nx_{nk})$$

As T reduces X_j, $A(T)$ also reduces X_j, $j = 1, 2, ..., k$. This implies that

$$y = R_1x_{1j} + R_2x_{2j} + ... + R_nx_nj = R_1\mid_{X_j} x_{1j} + R_2\mid_{X_j} x_{2j} + ... + R_n\mid_{X_j} x_{nj},$$

where $R_i\mid_{X_j} \in A(T_j)$ for all $i = 1, 2, ..., n$. Thus $(A(T_j), x_{1j}, x_{2j}, ..., x_{nj})$ is an a.f.s.m. on X_j.

This completes the proof of the theorem.

ACKNOWLEDGMENT. The authors are thankful to Prof. B.S. Yadav for the inspiration received in carrying out this study.

REFERENCES

1. LAMBERT, A., Strictly Cyclic Operator Algebras, Pacific J. Math. 39 (1971) 717-726.

2. LAMBERT, A., Spectral Properties of Strictly Cyclic Operator Algebras, Indiana Univ. Math. J.22 (1973) 959-963.

3. EMBRY, M.R., Strictly Cyclic Operator Algebras on Banach Space, Pacific J. Math 45 (1973) 445-452.

4. EMBRY, M.R., Maximal Invariant Subspaces of Strictly Cyclic Operator Algebras, Pacific. J. Math. 49 (1973) 45-50.

5. EMBRY, M.R., Self-adjoint Strictly Cyclic Operator Algebras, Pacific J. Math 52 (1974), 53-57.

6. BOLSTEIN, R., Strictly Cyclic Operators, Duke Math J. 40 (1973) 683-688.

7. HERRERO, D.A., Algebras de Operadores Que contieven Una Subalgebra de Multiplicidad Estucta Finite, Rev. de la Union Mat. Argentina 26(1972), 77-83

8. HERRERO, D.A., Operator Algebras of Finite-Strict Multiplicity, Indrana Univ. Math. J.22 (1972) 13-24.

9. ARORA, S.C. and BALA, PAWAN, Invariant Subspaces of Algebras of Finite Strict Multiplicity. (Communicated).

10. BARNES, BRUCE, Operators With a Strictly Cyclic Vector, Proc. Amer. Math. Soc. 41 (1973), 480-486.

11. HALPERN, H., Finite Sums of Irreducible Functionals on C∗-Algebras, Proc. Amer. Math. Soc. 18 (1967), 352-358.