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APPROXIMATION BY DOUBLE WALSH POLYNOMIALS

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ABSTRACT. We study the rate of approximation by rectangular partial sums, Cesàro means, and de la Vallée Poussin means of double Walsh-Fourier series of a function in a homogeneous Banach space X. In particular, X may be $L^p(I^2)$, where $1 \leq p < \infty$ and $I^2 = [0,1) \times [0,1)$, or $C_W(I^2)$, the latter being the collection of uniformly W-continuous functions on I^2. We extend the results by Watari, Fine, Yano, Jastrebova, Bljumin, Esfahanizadeh and Siddiqi from univariate to multivariate cases. As by-products, we deduce sufficient conditions for convergence in $L^p(I^2)$-norm and uniform convergence on I^2 as well as characterizations of Lipschitz classes of functions. At the end, we raise three problems.

KEY WORDS AND PHRASES. Walsh-Paley system, homogeneous Banach space, best approximation, W-continuity, modulus of continuity, Lipschitz class, rectangular partial sum, Cesàro mean, de la Vallée Poussin mean, Dirichlet kernel, Fejér kernel, convergence in L^p-norm, uniform convergence, saturation problem.
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1. INTRODUCTION.

We consider the Walsh orthonormal system $\{w_j(x): j \geq 0\}$ defined on the unit interval $I := [0,1)$ in the Paley enumeration (see [7]). To be more specific, let

$$r_0(x) := \begin{cases} 1 & \text{if} \quad x \in [0, 2^{-1}), \\ -1 & \text{if} \quad x \in [2^{-1}, 1), \end{cases}$$

$$r_0(x+1) := r_0(x),$$

$$r_j(x) := r_0(2^j x), \quad j \geq 1 \quad \text{and} \quad x \in I,$$

be the well-known Rademacher functions. For $k = 0$ set $w_0(x) := 1$, and if

$$k := \sum_{j=0}^{\infty} k_j 2^j, \quad k_j = 0 \quad \text{or} \quad 1,$$

is the dyadic representation of an integer $k \geq 1$, then set

$$w_k(x) := \prod_{j=0}^{\infty} [r_j(x)]^{k_j}.$$

We will study approximation by means of double Walsh polynomials in the norm of a homogeneous Banach space X of functions defined on the unit square $I^2 := [0,1) \times [0,1)$.

2. DOUBLE WALSH POLYNOMIALS AND MODULUS OF CONTINUITY.

We remind the reader that a double Walsh polynomial of order less than m in x and of order less than n in y is a two variable function of the form

$$P(x,y) := \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} a_{jk} w_j(x) w_k(y)$$

where m,n are positive integers and $\{a_{jk}\}$ is a double sequence of real (or complex) numbers. Denote by P_{mn} the collection of such Walsh polynomials and let

$$P := \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} P_{mn}.$$

The members of P are called double Walsh polynomials.

Denote by Σ_{mn} the finite σ-algebra generated by the collection of dyadic intervals of the form

$I_{mn}(j,k) := [j2^{-m},(j+1)2^{-m}) \times [k2^{-n},(k+1)2^{-n})$ where $0 \leq j < 2^m$, $0 \leq k < 2^n$, and $m,n \geq 0$. It is plain that the collection of Σ_{mn}-measurable functions defined on I^2 coincides with $P_{2^m,2^n}$. The so-called dyadic topology of I^2 is generated by the union of the Σ_{mn} for $m,n = 0,1,\ldots$.

The definition of a homogeneous Banach space on the circle group $T = [-\pi,\pi)$ is well-known (see Katznelson [6]). It is formulated on the dyadic group $I = [0,1)$, while using Walsh polynomials (see Butzer and Nessel [2] and also [8,pp. 154-155]). Following them, we say that a Banach space X of functions defined on I^2 with the norm $\|\cdot\|_X$ is homogeneous if $P \subseteq X \subseteq L^1(I^2)$ and if the following three properties hold:

(i) The norm of X dominates the $L^1(I^2)$-norm: for any $f \in X$

$$\|f\|_1 \leq \|f\|_X;$$

(ii) The norm of X is translation invariant: for any $(u,v) \in I^2$ and $f \in X$

$$\tau_{uv} f \in X \quad \text{and} \quad \|\tau_{uv} f\|_X = \|f\|_X$$

where τ_{uv} means the dyadic translation by u in the first variable and by v in the second one:

$$\tau_{uv} f(x,y) := f(x \dotplus u, y \dotplus v), \quad (x,y) \in I^2.$$

Here and in the sequel, $\dotplus$ denotes dyadic addition.

(iii) P is dense in X with respect to the norm $\|\cdot\|_X$, i.e., for any $f \in X$ and $\varepsilon > 0$ there exists a double Walsh polynomial $P \in P$ such that

$$\|P-f\|_X \leq \varepsilon.$$

We recall that the norm in $L^p(I^2), 1 \leq p < \infty$, is defined by

$$\|f\|_p := \left\{\int_0^1 \int_0^1 |f(x,y)|^p dxdy\right\}^{1/p},$$

while $C_W(I^2)$ is the collection of functions $f(x,y)$ that are uniformly continuous from the dyadic topology of I^2 to the usual topology of $\mathbf{R}$, and endowed with the "sup" norm:

$$\|f\|_\infty := \sup\{|f(x,y)| : (x,y)\in I^2\}.$$

Such a function f is called uniformly W-continuous.

Similarly to the univariate case (cf. [8, pp. 9-11]) if the periodic extension of a function $f(x,y)$ from I^2 to $\mathbf{R}^2$ with period 1 in both x and y is classically continuous, then f is also uniformly W-continuous on I^2.

It follows (cf. [8, p. 142] in the univariate case) that $L^p(I^2)$ is the closure of the collection P of double Walsh polynomials when using the norm $\|\cdot\|_p$, $1 \leq p < \infty$. Likewise (cf. [8, pp. 156-158]), $C_W(I^2)$ is the uniform closure of P, i.e., when using the norm $\|\cdot\|_\infty$.

The extension of [8, Lemma 1, p. 155] from I to I^2 is of basic importance in this paper.

LEMMA 1. For any $f,h\in X$ and $g\in L^1(I^2)$

$$\begin{aligned} &\left\|f*g-h\int_0^1\int_0^1 g(u,v)dudv\right\|_X \\ &\leq \int_0^1\int_0^1 \|\tau_{uv}f-h\|_X |g(u,v)|dudv \end{aligned} \tag{2.1}$$

where

$$(f*g)(x,y) := \int_0^1\int_0^1 f(x\dot{+}u,y\dot{+}v)g(u,v)dudv, \quad (x,y)\in I^2,$$

is the dyadic convolution of the functions f and g.

The proof of Lemma 1 is almost identical to that of the univariate lemma in [8, pp. 155-156]. We omit it.

Finally, we remind the reader that the (total) modulus of continuity of a function $f\in X$ is defined by

$$\omega_X(f;\delta_1,\delta_2) := \sup\{\|\tau_{uv}f-f\|_X : 0 \leq u < \delta_1,\ 0 \leq v < \delta_2\}$$

where $\delta_1,\delta_2 > 0$. By the Banach-Steinhaus theorem, for any $f\in X$

$$\lim_{u,v\to 0} \|\tau_{uv}f-f\|_X = 0,$$

and consequently,

$$\lim_{\delta_1,\delta_2\to 0} \omega_X(f;\delta_1,\delta_2) = 0.$$

For $\alpha,\beta > 0$, the Lipschitz class is defined by

$$\mathrm{Lip}(\alpha,\beta;X) := \{f\in X : \omega_X(f;\delta_1,\delta_2) = O(\delta_1^\alpha+\delta_2^\beta) \text{ as } \delta_1,\delta_2\to 0\}.$$

Unlike the classical case, $\mathrm{Lip}(\alpha,\beta;X)$ is not trivial when $\alpha > 1$ and/or $\beta > 1$ (cf. [8, p. 188]).

3. APPROXIMATION BY RECTANGULAR PARTIAL SUMS.

As is well-known, the measurement of the rate of approximation to a function $f\in X$ by polynomials in P_{mn} is defined by

$$E_{mn}(f;X):=\inf\{\|f-P\|_X : P\in P_{mn}\}.$$

Since P_{mn} is a finite dimensional subspace of X, for every $f\in X$ the infimum above is attained by some $P_{mn}\in P_{mn}$. Such a polynomial P_{mn} is called a best approximation of f in P_{mn}.

Given a function $f\in L^1(I^2)$, we form its double Walsh-Fourier series as follows

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk}w_j(x)w_k(y) \tag{3.1}$$

where the

$$a_{jk}:=\int_0^1\int_0^1 f(u,v)w_j(u)w_k(v)\,du\,dv, \quad j,k\geq 0,$$

are called double Walsh-Fourier coefficients of f. The rectangular partial sums of series (3.1) are defined by

$$S_{mn}(f;x,y):=\sum_{j=0}^{m-1}\sum_{k=0}^{n-1} a_{jk}w_j(x)w_k(y), \quad m,n\geq 1.$$

Now, the modulus of continuity gives sharp estimates to the rate of approximation by double Walsh polynomials $P\in P_{2^m,2^n}$ and by the rectangular partial sums $S_{2^m,2^n}(f)$.

THEOREM 1. For any $f\in X$ and $m,n\geq 0$,

$$\begin{aligned} 2^{-1}\omega_X(f;2^{-m},2^{-n}) &\leq E_{2^m,2^n}(f;X) \\ &\leq \|S_{2^m,2^n}(f)-f\|_X \leq \omega_X(f;2^{-m},2^{-n}). \end{aligned} \tag{3.2}$$

We note that the right inequality is the Walsh analogue of the classical Jackson inequality. The left-most inequality has no trigonometric analogue.

PROOF. As is well-known,

$$S_{mn}(f;x,y) = \int_0^1\int_0^1 f(x\dot{+}u,y\dot{+}v)D_m(u)D_n(v)\,du\,dv \tag{3.3}$$

where

$$D_m(u):=\sum_{j=0}^{m-1} w_j(u), \quad m\geq 1,$$

is the Walsh-Dirichlet kernel. We recall that the Paley lemma (see, e.g., [8, p 7]) says that

$$D_{2^m}(u) = \begin{cases} 2^m & \text{if } u\in[0,2^{-m}), \\ 0 & \text{if } u\in[2^{-m},1). \end{cases} \tag{3.4}$$

Now, by (2.1),

$$\|S_{2^m,2^n}(f)-f\|_X \leq \int_0^1\int_0^1 \|\tau_{uv}f-f\|_X D_{2^m}(u)D_{2^n}(v)\,dudv$$
$$= 2^{m+n}\int_0^{2^{-m}}\int_0^{2^{-n}} \|\tau_{uv}f-f\|_X dudv$$
$$\leq \omega_X(f,2^{-m},2^{-n}), \tag{3.5}$$

which is the third inequality in (3.2).

The second inequality in (3.2) is trivial.

We observe that for any polynomial $P\in P_{2^m,2^n}$ and $(u,v)\in I_{mn}(0,0)$ we have

$$P(x\dot{+}u,y\dot{+}v) = P(u,v).$$

Consequently, for such u,v

$$\tau_{uv}f-f = \tau_{uv}(f-P)-(f-P).$$

Now, let P be a best approximation to f in $P_{2^m,2^n}$. Then

$$\omega_X(f;2^{-m},2^{-n}) \leq 2\|f-P\|_X = 2E_{2^m,2^n}(f;X).$$

This is equivalent to the first inequality in (3.2).

The following corollary of Theorem 1 shows that the Lipschitz classes can be used to characterize functions by their rate of approximation by double Walsh polynomials.

COROLLARY 1. Let $f\in X$ and $\alpha,\beta > 0$. Then the following five statements are equivalent:

(a) $f\in\mathrm{Lip}(\alpha,\beta;X)$,

(b) $\|S_{2^m,2^n}(f)-f\|_X = O(2^{-m\alpha}+2^{-n\beta})$ as $m,n\to\infty$,

(c) $E_{2^m,2^n}(f;X) = O(2^{-m\alpha}+2^{-n\beta})$ as $m,n\to\infty$,

(d) $E_{jk}(f;X) = O(j^{-\alpha}+k^{-\beta})$ as $j,k\to\infty$,

(e) $\omega_X(f;2^{-m},2^{-n}) = O(2^{-m\alpha}+2^{-n\beta})$ as $m,n\to\infty$.

PROOF. According to Theorem 1, (a) implies (b) and (c).

By definition,

$$E_{jk}(f;X) \leq E_{il}(f;X) \quad \text{whenever} \quad j \geq i \quad \text{and} \quad k \geq l.$$

Consequently, if

$$2^m \leq j < 2^{m+1},\ 2^n \leq k < 2^{n+1}, \quad \text{and} \quad m,n \geq 0, \tag{3.6}$$

then

$$E_{2^{m+1},2^{n+1}}(f;X) \leq E_{jk}(f;X) \leq E_{2^m,2^n}(f;X). \tag{3.7}$$

Hence it follows that (c) and (d) are equivalent.

By Theorem 1 and (3.7), (d) implies (e).

Finally, the fact that $\omega_X(f;\delta_1,\delta_2)$ decreases as either δ_1 or δ_2 decreases shows that (e) and (a) are equivalent.

On closing, we note that Theorem 1 and Corollary 1 are the multivariate extensions of the corresponding results by Watari [9], proved for the cases $X = C_W(I)$ and $L^p(I), 1 \leq p < \infty$.

4. APPROXIMATION BY CESÀRO MEANS.

As is well-known, the first arithmetic means or Cesàro means of series (3.1) are defined by

$$\sigma_{mn}(f;x,y) := \frac{1}{mn} \sum_{j=1}^{m} \sum_{k=1}^{n} S_{jk}(f;x,y), \quad m,n \geq 1.$$

It follows from (3.3) that

$$\sigma_{mn}(f;x,y) = \int_0^1 \int_0^1 f(x\dot{+}u, y\dot{+}v) K_m(u) K_n(v)\,du\,dv \tag{4.1}$$

where

$$K_m(u) := \frac{1}{m} \sum_{j=1}^{m} D_j(u)$$

is the Walsh-Fejér kernel. This kernel has the remarkable property of quasi-positiveness:

$$\|K_m\|_1 := \int_0^1 |K_m(u)|\,du \leq 2, \; m \geq 1,$$

first proved by Yano [10]. By (2.1), we conclude that for any $f \in X$

$$\|\sigma_{mn}(f)\|_X \leq \|K_m\|_1 \|K_n\|_1 \|f\|_X \leq 4\|f\|_X. \tag{4.2}$$

We estimate the rate of convergence when a function is approximated by the Cesàro means of its double Walsh-Fourier series.

THEOREM 2. For any $f \in X$ and $j,k \geq 1$,

$$\|\sigma_{jk}(f) - f\|_X \leq 6 \sum_{i=0}^{m} \sum_{l=0}^{n} 2^{i+l-m-n} \omega_X(f; 2^{-i}, 2^{-l}) \tag{4.3}$$

where m and n are defined in (3.6).

The next two corollaries are immediate consequences of Theorem 2.

COROLLARY 2. (i) If $f \in L^p(I^2)$ for some $1 \leq p < \infty$, then the Cesàro means $\sigma_{jk}(f)$ of its double Walsh-Fourier series converge to f in L^p-norm.

(ii) If $f \in C_W(I^2)$, then the $\sigma_{jk}(f)$ converge to f uniformly on I^2.

In statement (i), the case $p = 1$ in really interesting. In Section 6, we will prove that, in the cases when $1 < p < \infty$, even the rectangular partial sums $S_{jk}(f)$ converge to f in L^p-norm (see Theorem 5 below). Statement (ii) is the multivariate extension of the corresponding result by Fine [4].

COROLLARY 3. If $f \in \mathrm{Lip}(\alpha,\beta;X)$ for some $\alpha,\beta > 0$, then

$$\|\sigma_{jk}(f) - f\|_X = \begin{cases} O(j^{-\alpha} + k^{-\beta}) & \text{if } 0 < \alpha,\beta < 1, \\ O(j^{-1} \log j + k^{-\beta}) & \text{if } 0 < \beta < 1 = \alpha, \\ O(j^{-1} + k^{-\beta}) & \text{if } 0 < \beta < 1 < \alpha, \\ O(j^{-1} \log j + k^{-1} \log k) & \text{if } \alpha = \beta = 1, \\ O(j^{-1} + k^{-1} \log k) & \text{if } 1 = \beta < \alpha, \\ O(j^{-1} + k^{-1}) & \text{if } 1 < \alpha,\beta. \end{cases} \tag{4.4}$$

We note that Corollary 3 is also the multivariate extension of the corresponding results by Yano [11] (proved for $0 < \alpha < 1$ and $1 \leq p \leq \infty$) and by Jastrebova [5] (proved for $\alpha = 1$ and $p = \infty$).

PROOF OF THEOREM 2. Keeping (3.6) in mind, we may write

$$\sigma_{jk}(f)-f = \frac{2^{m+n}}{jk} \left(\frac{1}{2^{m+n}} \sum_{i=1}^{2^m} \sum_{l=1}^{2^n} S_{il}(f)-f \right)$$

$$+ \frac{1}{jk}\left\{ \sum_{i=2^m+1}^{j} \sum_{l=1}^{2^n} + \sum_{i=1}^{2^m} \sum_{l=2^n+1}^{k} + \sum_{i=2^m+1}^{j} \sum_{l=2^n+1}^{k} \right\}(S_{il}(f)-S_{2^m,2^n}(f))$$

$$+\left(1-\frac{2^{m+n}}{jk} \right) (S_{2^m,2^n}(f)-f)$$

$$= \frac{2^{m+n}}{jk}(\sigma_{2^m,2^n}(f)-f)+\sigma_{mn}(f-S_{2^m,2^n}(f)) + \left(1-\frac{2^{m+n}}{jk} \right) (S_{2^m,2^n}(f)-f).$$

Hence, by the triangle inequality and (4.1),

$$\|\sigma_{jk}(f)-f\|_X \le \|\sigma_{2^m,2^n}(f)-f\|_X+5\|S_{2^m,2^n}(f)-f\|_X.$$

Consequently, by Theorem 1,

$$\|\sigma_{jk}(f)-f\|_X \le \|\sigma_{2^m,2^n}(f)-f\|_X+5\omega_X(f;2^{-m},2^{-n}). \tag{4.5}$$

Now, we estimate the first quantity on the right-hand side of (4.5). To this end, we recall the representation

$$K_{2^m}(u) = 2^{-1}\left\{2^{-m}D_{2^m}(u)+ \sum_{i=0}^{m} 2^{i-m}D_{2^m}(u\dot{+}2^{-i-1})\right\}, \quad u\in I$$

(see, e.g., [8, p. 46, relation (iii)]). By (3.4), for $0 \le i < m$

$$D_{2^m}(u\dot{+}2^{-i-1}) = \begin{cases} 2^m & \text{if} \quad u\in[2^{-i-1},2^{-i-1}+2^{-m}), \\ 0 & \text{otherwise;} \end{cases}$$

and for i = m

$$D_{2^m}(u\dot{+}2^{-m-1}) = D_{2^m}(u) = \begin{cases} 2^m & \text{if} \quad u\in[0,2^{-m}), \\ 0 & \text{otherwise.} \end{cases}$$

In particular, it follows that $K_{2^m}(u) \ge 0$ for all $u\in I$.

Similarly to (3.5), we apply again (2.1) and then by an elementary reasoning we obtain that

$$\|\sigma_{2^m,2^n}(f)-f\|_X \le \int_0^1 \int_0^1 \|\tau_{uv}f-f\|_X K_{2^m}(u)K_{2^n}(v)\,dudv$$

$$= \left\{\int_0^{2^{-m}} \int_0^{2^{-n}} + \sum_{i=0}^{m-1} \int_{2^{-i-1}}^{2^{-i}} \int_0^{2^{-n}} + \sum_{l=0}^{n-1} \int_0^{2^{-m}} \int_{2^{-l-1}}^{2^{-l}} \right.$$

$$\left. + \sum_{i=0}^{m-1} \sum_{l=0}^{n-1} \int_{2^{-i-1}}^{2^{-i}} \int_{2^{-l-1}}^{2^{-l}} \right\}\|\tau_{uv}f-f\|_X K_{2^m}(u)K_{2^n}(v)\,dudv$$

$$\le \omega_X(f;2^{-m},2^{-n})+\sum_{i=0}^{m-1} 2^{i-m-1}\omega_X(f;2^{-i},2^{-n})$$

$$+\sum_{l=0}^{n-1} 2^{l-n-1}\omega_X(f;2^{-m},2^{-l})$$

$$-\sum_{i=0}^{m-1}\sum_{l=0}^{n-1} 2^{i-m-1}2^{l-n-1}\omega_X(f;2^{-i},2^{-l})$$

$$\leq \sum_{i=0}^{m}\sum_{l=0}^{n} 2^{i+l-m-n}\omega_X(f;2^{-i},2^{-l}). \tag{4.6}$$

Combining (4.5) and (4.6) yields (4.3).

5. APPROXIMATION BY DE LA VALLÉE POUSSIN MEANS.

By Corollaries 1 and 3, the rate of approximation by $\sigma_{jk}(f)$ is as good as by $S_{2^m,2^n}(f)$ if $f\in\text{Lip}(\alpha,\beta;X)$ for some $0 < \alpha,\beta < 1$, where m and n are defined by (3.6). However, the $\sigma'_{jk}s$ are not projections from X onto P_{jk}. These two important properties are satisfied by the de la Vallée Poussin means of series (3.1) defined by

$$V_{mn}(f;x,y) := \frac{1}{mn}\sum_{j=m+1}^{2m}\sum_{k=n+1}^{2n} S_{jk}(f;x,y), \quad m,n \geq 1.$$

THEOREM 3. For any $f\in X$ and $m,n \geq 1$,

$$\| V_{mn}(f)-f\|_X \leq 37E_{mn}(f;X). \tag{5.1}$$

We note that in the univariate case, Bljumin [1], Esfahanizadeh and Siddiqi [3] studied de la Vallée Poussin means and obtained an inequality whose multivariate extension is (5.1).

PROOF. A routine computation shows that

$$\begin{aligned} V_{mn}(f;x,y) &= S_{mn}(f;x,y) \\ &+ 2\sum_{j=m}^{2m-1}\sum_{k=0}^{n-1}\left(1-\frac{j}{2m}\right) a_{jk}w_j(x)w_k(y) \\ &+ 2\sum_{j=0}^{m-1}\sum_{k=n}^{2n-1}\left(1-\frac{k}{2n}\right) a_{jk}w_j(x)w_k(y) \\ &+ 4\sum_{j=m}^{2m-1}\sum_{k=n}^{2n-1}\left(1-\frac{j}{2m}\right)\left(1-\frac{k}{2n}\right) a_{jk}w_j(x)w_k(y). \end{aligned}$$

Hence it follows immediately that for any $P\in P_{mn}$

$$V_{mn}(P;x,y) = P(x,y). \tag{5.2}$$

On the other hand, it is easy to check that

$$V_{mn}(f) = 4\sigma_{2m,2n}(f)-2\sigma_{2m,n}(f)-2\sigma_{m,2n}(f)+\sigma_{mn}(f).$$

Consequently, by (4.2), for any $f\in X$ and $m,n \geq 1$ we have

$$\| V_{mn}(f)\|_X \leq 36\| f\|_X. \tag{5.3}$$

Now, let P be a best approximation to f in P_{mn}. Then, combining (5.2) and (5.3) yields

$$\|V_{mn}(f)-f\|_X \leq \|V_{mn}(f-P)\|_X+\|P-f\|_X$$
$$\leq 37\|P-f\|_X = 37E_{mn}(f;X).$$

6. ESTIMATION AND SATURATION PROBLEMS.

(A) Theorem 1 says that the rate of approximation by the rectangular partial sums $S_{2^m,2^n}(f)$ of the Walsh-Fourier series (3.1) is no worse than that by double Walsh polynomials from $P_{2^m,2^n}$ at all. As to approximation by $S_{mn}(f)$, we can ensure only a weaker rate in general.

THEOREM 4. For any $f\in X$ and $m,n \geq 1$,

$$\|S_{mn}(f)-f\|_X \leq (1+\|D_m\|_1\|D_n\|_1)E_{mn}(f;X). \tag{6.1}$$

This can be proved in a routine way. For the reader's convenience, we sketch it.

PROOF. Let P be a best approximation to f in P_{mn}. Since $S_{mn}(P) = P$, we may write that

$$\|S_{mn}(f)-f\|_X \leq \|S_{mn}(f-P)\|_X+\|P-f\|_X. \tag{6.2}$$

Taking into account (3.3), (2.1), and the fact that $\|\cdot\|_X$ is translation invariant gives that

$$\|S_{mn}(f-P)\|_X \leq \int_0^1\int_0^1 \|\tau_{uv}(f-P)\|_X|D_m(u)D_n(v)|\,dudv$$
$$= \|f-P\|_X\|D_m\|_1\|D_n\|_1. \tag{6.3}$$

Now, (6.1) follows from (6.2) and (6.3).

We note that

$$\|D_m\|_1 = O(\log m)$$

and this estimate is sharp (see, e.g. [8, p. 35]). In spite of this fact, estimate (6.1) can be essentially improved in the particular case when $X = L^p(I^2)$, $1 < p < \infty$. We will write $\|\cdot\|_p'$ instead of $\|\cdot\|_{L^p(I^2)}$.

THEOREM 5. For any $1 < p < \infty$, there exists a constant $\tilde{K}_p$ such that for any $f\in L^p(I^2)$ and $m,n \geq 1$ we have

$$\|S_{mn}(f)-f\|_p \leq \tilde{K}_pE_{mn}(f;L^p(I^2)). \tag{6.4}$$

Theorem 5 is ultimately a consequence of the following result by Paley [7]: For any $1 < p < \infty$, there exists a constant K_p such that for any $g\in L^p(I)$ and $m \geq 1$ we have

$$\|S_m(g)\|_p \leq K_p\|g\|_p \tag{6.5}$$

where this time

$$S_m(g;x) := \sum_{j=0}^{m-1} \left(\int_0^1 g(u)w_j(u)\,du \right) w_j(x) \quad \text{and} \quad \|g\|_p := \left\{\int_0^1 |g(x)|^p dx\right\}^{1/p}.$$

On the basis of (6.5) we will prove the following

LEMMA 2. For any $f\in L^p(I^2)$, $1 < p < \infty$, and $m,n \geq 1$,

$$\|S_{mn}(f)\|_p \le K_p^2 \|f\|_p . \tag{6.6}$$

PROOF. We may consider $f(x,y)$ as a function of x for each fixed y denoted by $g_y(x):= f(x,y)$. Observe that if $f\in L^p(I^2)$, then $g_y\in L^p(I)$ for almost all $y\in I$ and

$$S_{mn}(f;x,y) = S_n(S_m(g_y;x);y), \quad m,n \ge 1.$$

Furthermore, if $f\in L^p(I^2)$, then

$$G_{m,x}(y):= S_m(g_y;x) = \sum_{j=0}^{m-1} \left(\int_0^1 g_y(u)w_j(u)du \right) w_j(x)\in L^p(I)$$

for all $m \ge 1$ and for almost all $x\in I$.

Now, applying Fubini's theorem three times and the univariate inequality (6.5) twice provides (6.6) as follows

$$\int_0^1\int_0^1 |S_{mn}(f;x,y)|^p dxdy$$

$$= \int_0^1 \left\{ \int_0^1 \left| \sum_{k=0}^{n-1} \left(\int_0^1 G_{m,x}(v)w_k(v)dv \right) w_k(y) \right|^p dy \right\} dx$$

$$\le \int_0^1 K_p \left\{ \int_0^1 |G_{m,x}(y)|^p dy \right\} dx$$

$$= K_p \int_0^1 \left\{ \int_0^1 \left| \sum_{j=0}^{m-1} \left(\int_0^1 g_y(u)w_j(u)du \right) w_j(x) \right|^p dx \right\} dy$$

$$\le K_p \int_0^1 K_p \left\{ \int_0^1 |g_y(x)|^p dx \right\} dy$$

$$= K_p^2 \int_0^1\int_0^1 |f(x,y)|^p dxdy.$$

After these preliminaries, the proof of Theorem 5 is identical with that of Theorem 4, except that we use (6.6) instead of (6.3). In this way, we arrive at (6.4) with $\tilde{K}_p:= 1+K_p^2$.

Obviously, Theorem 5 implies the following

COROLLARY 4. If $f\in L^p(I^2)$ for some $1 < p < \infty$, then the rectangular partial sums $S_{mn}(f)$ of its double Walsh-Fourier series converge to f in L^p-norm.

Nevertheless, it seems to be very likely that estimate (6.1) is the best possible in general.

PROBLEM 1. Show that, in the cases when $X = L^1(I^2)$ or $C_W(I^2)$, there exists a function $f\in X$ such

$$\limsup_{m,n\to\infty} \frac{\|S_{mn}(f)-f\|_X}{\log m \log n} > 0.$$

(B) We guess that Corollary 3 is also the best possible in the above sense. For example, we formulate this in connection with the fourth estimate in (4.4).

PROBLEM 2. Show that, in the cases when $X = L^1(I^2)$ or $C_W(I^2)$, there exists a function $f \in \mathrm{Lip}(1,1;X)$ such that

$$\limsup_{m,n\to\infty} \frac{\|\sigma_{mn}(f)-f\|_X}{m^{-1}\log m + n^{-1}\log n} > 0.$$

In the univariate case, the corresponding result was proved by Jastrebova [5] with "lim" instead of "lim sup".

(C) Finally, we discuss the so-called saturation problem. We begin with the observation that the rate of approximation by the Cesàro means $\sigma_{mn}(f)$ to functions $f \in \mathrm{Lip}(\alpha,\beta)$ is not improved as α and β increase beyond 1. Indeed, the following is true.

THEOREM 6. If for some $f \in X$

$$\|\sigma_{2^n,2^n}(f)-f\|_X = o(2^{-n}) \quad \text{as} \quad n \to \infty, \tag{6.7}$$

then f is constant.

PROOF. Since

$$E_{2^n,2^n}(f;X) \leq \|\sigma_{2^n,2^n}(f)-f\|_X,$$

by hypothesis and Theorem 1, we have

$$\|S_{2^n,2^n}(f)-f\|_X = o(2^{-n}) \quad \text{as} \quad n \to \infty. \tag{6.8}$$

Taking into account that

$$2^n(S_{2^n,2^n}(f;x,y)-\sigma_{2^n,2^n}(f;x,y)) = \sum_{j=0}^{2^n-1}\sum_{k=0}^{2^n-1}(j+k-2^{-n}jk)a_{jk}w_j(x)w_k(y),$$

by (6.7) and (6.8), we conclude that

$$\lim_{n\to\infty}\left\|\sum_{j=0}^{2^n-1}\sum_{k=0}^{2^n-1}(j+k-2^{-n}jk)a_{jk}w_j(x)w_k(y)\right\|_X = 0.$$

Since $\|\cdot\|_1 \leq \|\cdot\|_X$, it follows that

$$|(j_0+k_0)a_{j_0,k_0}|$$

$$= \lim_{n\to\infty}\left|\int_0^1\int_0^1 w_{j_0}(x)w_{k_0}(y)\sum_{j=0}^{2^n-1}\sum_{k=0}^{2^n-1}(j+k-2^{-n}jk)a_{jk}w_j(x)w_k(y)\,dx\,dy\right|$$

$$\leq \lim_{n\to\infty}\left\|\sum_{j=0}^{2^n-1}\sum_{k=0}^{2^n-1}(j+k-2^{-n}jk)a_{jk}w_j(x)w_k(y)\right\|_1 = 0$$

for all $j_0,k_0 \geq 0$ such that $\max(j_0,k_0) \geq 1$. This implies that $a_{j_0,k_0} = 0$ for all such pairs j_0,k_0, and therefore, $f = a_{00}$ is constant.

PROBLEM 3. How can one characterize those functions $f \in X$ such that

$$\|\sigma_{jk}(f)-f\|_X = O(j^{-1}+k^{-1}) \ ? \tag{6.9}$$

This is not known even in the univariate case. We conjecture that, in the special case when $j = k = 2^n$, $X = C_W(I^2)$ or $L^p(I^2)$ for some $1 \leq p < \infty$, we have

$$\|\sigma_{2^n,2^n}(f)-f\|_X = O(2^{-n})$$

if and only if

$$\sum_{i=0}^{n}\sum_{l=0}^{n} 2^{i+l}\omega(f;2^{-i},2^{-l}) = O(2^n).$$

The "if" part follows from (4.6). The proof (or disproof) of the "only if" part is a problem.

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