Internat. J. Math. & Math. Sci.
VOL. 15 NO. 2 (1992) 241-254

LINEAR FUNCTIONALS ON ORLICZ SEQUENCE SPACES WITHOUT LOCAL CONVEXITY

MARIAN NOWAK

Institute of Mathematics
A. Mickiewicz University
Matejki 48/49, 60-769 Poznań
Poland

(Received March 8, 1990)

ABSTRACT. The general form of continuous linear functionals on an Orlicz sequence space l^ϕ (non-separable and non-locally convex in general) is obtained. It is proved that the space h^ϕ is an M-ideal in l^ϕ.

KEY WORDS AND PHRASES. Orlicz sequence spaces, Köthe dual, Riesz spaces, Mackey topologies, modular spaces, and M-ideals.
1991 AMS SUBJECT CLASSIFICATION CODE. Primary 46 E 30.

INTRODUCTION. The general form of continuous linear functionals on an Orlicz space L^ϕ, defined by a convex Orlicz function ϕ has been found by Ando [2] (for ϕ being an N-function and for a finite measure space) and by Rao [21], Fernandez [7] (for ϕ being a Young function and for a general measure space).

In this paper we describe the dual space $(l^\phi)^*$ of an Orlicz sequence space l^ϕ defined by an arbitrary Orlicz function ϕ (not necessarily convex) such that $\phi(u)/u \to \infty$ as $u \to \infty$. For this purpose we shall first use the description of the Mackey topology τ_ϕ of l^ϕ, obtained by Kalton [8], when ϕ satisfies the Δ_2-condition at 0, and by Drewnowski and Nawrocki [5], in general. The Mackey topology τ_ϕ is normable and we consider two natural norms on l^ϕ which generate τ_ϕ. Thus we can define two corresponding norms in $(l^\phi)^*$. Moreover, we consider l^ϕ from the point of view of the theory of modular spaces (see [15], [16], [17]). We investigate the conjugate modular (in the sense of Nakano [17]) on $(l^\phi)^*$ and consider two other norms on $(l^\phi)^*$ defined in a natural way by the conjugate modular. It is well-known that $(l^\phi)^* = (l^\phi)^\sim_n + (l^\phi)^\sim_s$, where $(l^\phi)^\sim_n$ and $(l^\phi)^\sim_s$ denote the sets of all order continuous and singular linear functionals on l^ϕ respectively. We first show that the Köthe dual $(l^\phi)^x$ of l^ϕ coincides with the Orlicz sequence space l^{ϕ^*}, where ϕ^* denotes the complementary function of φ in the sense of Young. Thus we obtain the corresponding characterization of $(l^\phi)^\sim_n$. Next, we prove that the conjugate modular and all four norms defined on $(l^\phi)^*$ coincide on $(l^\phi)^\sim_s$. Following the idea of [2] we construct a Riesz isometric isomorphism of $(l^\phi)^\sim_s$ onto some Riesz subspace $B_\phi(\mathrm{N})$ (dependent on ϕ) of the Banach lattice $ba(\mathrm{N})$ of all real-valued bounded finitely additive set functions on N. We prove that there exists an isometric isomorphism of the Banach space $\left((l^\phi)^*, \|\cdot\|^*_\phi\right)$ (for the definition of the norm $\|\cdot\|^*_\phi$ see section 2) onto the Banach space $l^{\phi^*} \times B_\phi(N)$ given by the mapping $f \to (y,\nu)$ such that $f(x) = \sum_{i=1}^{\infty} x(i)y(i) + \int x \, d\nu$ for all $x \in l^\phi$ and $\|f\|^*_\phi = \|y\|_{\phi^*} + |\nu|(N)$. From this it follows that h^ϕ (the ideal of elements of absolutely continuous F-norm on l^ϕ) is an M-ideal of l^ϕ (see [3, definition 2.1]). As an application, we obtain that every continuous linear function on h^ϕ has the unique norm preserving extension to l^ϕ.

1. Preliminaries. For terminology concerning locally solid Riesz spaces we refer to [1] and [14]. For a Riesz space $(E, \geq)$ let $E^+ = \{u \in E \ : \ u \geq 0\}$ (the positive cone of E). By N we will denote the set of all natural numbers. Denote by ω the space of all real-valued sequences. For the sequence x, $x(i)$ means the

i-th coordinate of x, and we shall denote by $x^{(n)}$ the n-th section of x (that is $x^{(n)}(i) = x(i)$ for $i \leq n$, $x^{(n)}(i) = 0$ for $i > n$). For a subset A of N we will denote by x_A the sequence such that $x_A(i) = x(i)$ for $i \in A$ and $x_A(i) = 0$ for $i \notin A$. If f is a linear functional on a subspace X of ω, we will denote by f_A the functional defined as: $f_A(x) = f(x_A)$ for $x \in X$. It is known that ω is a super Dedekind complete Riesz space under the ordering $x \leq y$ whenever $x(i) \leq y(i)$ for $i \in$ N.

Now we recall some terminology concerning Orlicz sequence spaces (see [11], [12], [22], and [25]).

By <u>an Orlicz function</u> ϕ we mean a function ϕ: $[0,\infty) \to [0,\infty)$ which is non-decreasing, continuous for $u \geq 0$ and $\phi(u) = 0$ iff $u = 0$. Throughout this paper we shall assume that φ satisfies the following condition: $\phi(u)/u \to \infty$ as $u \to \infty$. Every Orlicz function ϕ determines the functional $\rho_\phi : \omega \to [0,\infty]$ defined by the formula:

$$\rho_\phi(x) = \sum_{i=1}^{\infty} \phi(|x(i)|).$$

Then $1^\phi = \{x \in \omega : \rho_\phi(\lambda x) < \infty \text{ for some } \lambda > 0\}$ is called <u>an Orlicz sequence space</u> defined by ϕ. The space 1^ϕ is an ideal of ω and the functional ρ_ϕ restricted to 1^ϕ <u>is an orthogonal additive modular</u>, i.e., ρ_ϕ satisfies the following conditions:

(1) $\rho_\phi(x) = 0$ iff $x = 0$.

(2) $\rho_\phi(x_1) \leq \rho_\phi(x_2)$ if $|x_1| \leq |x_2|$.

(3) $\rho_\phi(\lambda x) \to 0$ if $\lambda \to 0$.

(4) $\rho_\phi(x_1 + x_2) = \rho_\phi(x_1) + \rho_\phi(x_2)$ if $|x_1| \wedge |x_2| = 0$.

These conditions imply that $\rho_\phi(x_1 \vee x_2) \leq \rho_\phi(x_1) + \rho_\phi(x_2)$ for $x_1, x_2 \geq 0$. Moreover, ρ_ϕ satisfies the following axiom of completeness (see [15]):

(C) If $x_n \geq 0$ for $n = 1,2,\dots$ and $\sum_{n=1}^{\infty} \rho_\phi(x_n) < \infty$, then there exists $y \in 1^\phi$ such that $y = \sup x_n$ and $\rho_\phi(y) \leq \sum_{n=1}^{\infty} \rho_\phi(x_n)$.

If ϕ is a convex Orlicz function, then the modular ρ_ϕ is convex, i.e.,

$$\rho_\phi(ax_1 + bx_2) \leq a\rho_\phi(x_1) + b\rho_\phi(x_2) \text{ for } a,b \geq 0 \text{ with } a + b = 1.$$

In 1^ϕ the complete Riesz F-norm $\|\cdot\|_\phi$ can be defined by

$$|x|_\phi = \inf\{\lambda > 0 : \rho_\phi(x/\lambda) \leq \lambda\}.$$

We shall denote by τ_ϕ the topology of the F-norm $|\cdot|_\phi$. Let $h^\phi = \{x \in 1^\phi : \rho_\phi(\lambda x) < \infty \text{ for all } \lambda > 0\}$. Then h^ϕ is the ideal of elements of absolutely continuous F-norm $|\cdot|_\phi$ on 1^ϕ.

We say that ϕ satisfies <u>the Δ_2-condition at 0</u>, whenever $\limsup_{u \to 0} \phi(2u)/\phi(u) < \infty$. It is known that $1^\phi = h^\phi$ (i.e. 1^ϕ is separable) iff ϕ satisfies the Δ_2-condition at 0.

We say that two Orlicz functions ϕ and ψ are equivalent at 0, in symbols $\phi \sim \psi$, if there exist positive numbers a,b,c,d and $u_0 > 0$ such that $a\phi(bu) \leq \psi(u) \leq c\phi(du)$ for $0 \leq u \leq u_0$. It is well-known that if $\phi \sim \psi$ then $1^\phi = 1^\psi$ and $\tau_\phi = \tau_\psi$. Moreover, the space $(1^\phi, \tau_\phi)$ is locally convex iff there exists a convex Orlicz function ψ such that $\phi \sim \psi$ (see [25], Theorem 3.1.5]. Separable Orlicz sequence spaces without local convexity have been investigated in detail by Kalton [8]. For examples of non-separable and non-locally convex Orlicz sequence spaces see [5].

We denote by p_ϕ the Minkowski functional of the absolutely convex absorbing subset $k^\phi = \{x \in \omega : \rho_\phi(x) < \infty\}$ of 1^ϕ. Thus

$$p_\phi(x) = \inf\{\lambda > 0 : \rho_\phi(x/\lambda) < \infty\}$$

for all $x \in 1^\phi$, $p_\phi(x) \leq |x|_\phi$ for $x \in 1^\phi$, and $h^\phi = \ker\, p_\phi$.

2. Norms on the dual space $(l^\phi)^*$ of l^ϕ. In this section we define in two different ways some natural norms on $(l^\phi)^*$. For this purpose we shall first use the description of the Mackey topology of (l^ϕ, τ_ϕ) given in [5], and next, we apply the Nakano's theory of conjugate modulars [17].

Let us put

$$\phi^*(v) = \sup\{uv - \phi(u) : u \geq 0\} \text{ for } v \geq 0.$$

Then ϕ^* will be called the function complementary to ϕ in the sense of Young. It is seen that ϕ^* is a convex function, taking only finite values, and $\phi^*(0) = 0$. This means that ϕ^* is a Young function (see [12], [13], [26]). The additional properties of ϕ^* are included in the following

LEMMA 2.1. (a) If $\liminf_{u \to 0} \phi(u)/u = 0$, then ϕ^* vanishes only at 0 and $\lim_{v \to 0} \phi^*(v)/v = 0$, $\lim_{v \to \infty} \phi^*(v)/v = \infty$ (i.e. ϕ^* is an N-function in the sense of [11]).

(b) If $\liminf_{u \to 0} \phi(u)/u > 0$, then ϕ^* vanishes near zero and $\lim_{v \to \infty} \phi^*(v)/v = \infty$ (i.e. $l^{\phi^*} = l^\infty$).

PROOF. (a) We can easily verify that $\phi^*(v) > 0$ for $v > 0$. In the same way as in [4, §2] we can show that $\lim_{v \to 0} \phi^*(v)/v = 0$ and $\lim_{v \to \infty} \phi^*(v)/v = \infty$.

(b) We shall show that there exists $v_0 > 0$ such that $\phi^*(v) = 0$ for $0 \leq v \leq v_0$, and $\phi^*(v) > 0$ for $v > v_0$. Indeed, since $\liminf_{u \to 0} \phi(u)/u > 0$ there exist numbers $v' > 0$ and $u' > 0$ such that $uv' \leq \phi(u)$ for $0 \leq u \leq u'$, and since $\lim_{u \to \infty} \phi(u)/u = \infty$ (by our assumption) there exists a number $u'' > 0$ with $u'' > u'$ such that $u \leq \phi(u)$ for $u \geq u''$. Taking $v'' > 0$ such that $1/v'' = \sup\{u/\phi(u) : u' \leq u \leq u''\}$, we have $uv'' \leq \phi(u)$ for $u' \leq u \leq u''$. Then for $v_1 = \min(1, v', v'')$ we get $uv_1 \leq uv' \leq \phi(u)$ for $u \geq u''$, $uv_1 \leq uv'' \leq \phi(u)$ for $u' \leq u \leq u''$, and $uv_1 \leq u \leq \phi(u)$ for $u \geq u''$. Hence $uv_1 - \phi(u) \leq 0$ for $u \geq 0$, so that $\phi^*(v_1) = 0$. On the other hand, there exists a number $v_2 > 0$ such that $\phi^*(v_2) > 0$. Since ϕ^* is convex, there exists a number $v_0 > 0$ such that $\phi^*(v) = 0$ for $0 \leq v \leq v_0$, and $\phi^*(v) > 0$ for $v > v_0$. Moreover, as in [4, §2] we can show that $\lim_{v \to \infty} \phi^*(v)/v = \infty$.

For an Orlicz function ϕ we shall denote by $\hat{\phi}$ the convex minorant of ϕ in a neighborhood of 0, i.e., $\hat{\phi}$ is the largest Orlicz function such that $\hat{\phi}(u) \leq \phi(u)$ for $u \geq 0$, and $\hat{\phi}$ is convex on the interval $[0,1]$ (see [8, p. 255]).

Moreover, let us put

$$\bar{\phi}(u) = (\phi^*)^*(u) \quad \text{for} \quad u \geq 0.$$

It is seen that $\bar{\phi}$ is a convex Orlicz function such that $\lim_{u \to \infty} \bar{\phi}(u)/u = \infty$. The relation between $\hat{\phi}$ and $\bar{\phi}$ is described by

LEMMA 2.2. We have $\hat{\phi} \sim \bar{\phi}$ and $\bar{\phi}(u) \leq \phi(u)$ for $u \geq 0$.

PROOF. First, we shall show that $\bar{\phi}(u) \leq \phi(u)$ for $u \geq 0$. Indeed, since $\lim_{v \to \infty} \phi^*(v)/v = \infty$, for every $u > 0$ there exists $v_u > 0$ such that $\bar{\phi}(u) + \phi^*(v_u) = uv_u$. But $uv_u \leq \phi(u) + \phi^*(v_u)$; hence $\bar{\phi}(u) \leq \phi(u)$ for $u \geq 0$. In [18, Lemma 2.1] it is proved that $\hat{\phi} \sim \bar{\phi}$ whenever $\liminf_{u \to 0} \phi(u)/u = 0$. Now assume that $\liminf_{u \to 0} \phi(u)/u > 0$. We can check that $\hat{\phi} \sim \chi_1$, where $\chi_1(u) = u$ for $u \geq 0$ (see [18]). It suffices to show that $\bar{\phi} \sim \chi_1$. In view of Lemma 2.1 there exists a number $v_0 > 0$ such that $\phi^*(v) = 0$ for $0 \leq v \leq v_0$, and $\phi^*(v) \geq 0$ for $v > v_0$. Moreover, since $\lim_{v \to \infty} \phi^*(v)/v = \infty$, for every $u > 0$ there exists $v_u > v_0$ such that $uv - \phi^*(v) < 0$ for $v > v_u$. Hence, for every $u > 0$, $\bar{\phi}(u) = \max(uv_0, \sup\{uv - \phi^*(v) : v_0 \leq v \leq v_u\})$. But $\sup\{uv - \phi^*(v) : v_0 \leq v \leq v_u\} = uv' - \phi^*(v')$ for some v' with $v_0 \leq v' \leq v_u$. Assuming that $v_0 < v'$, we obtain that $\bar{\phi}(u) = uv_0$ for $0 \leq u \leq u_0 = \phi^*(v')/(v' - v_0)$, and thus $\bar{\phi} \sim \chi_1$.

For a topological vector space (E,ξ) we shall denote by $(E,\xi)^*$ its topological dual. We shall denote by $(l^\phi)^*$ the dual space of (l^ϕ,τ_ϕ).

Let us recall that the <u>Mackey topology</u> of (E,ξ) is the finest locally convex topology τ which produces the same continuous linear functionals as the original topology ξ. If (E,ξ) is an F-space then τ is the finest locally convex topology on E which is weaker than ξ (see [24]).

Kalton [8] has showed that the Mackey topology τ_ϕ of a separable Orlicz sequence space l^ϕ coincides with the topology $\tau_{\bar\phi|_{l^\phi}}$ induced from $l^{\bar\phi}$. For an arbitrary l^ϕ, the Mackey topology τ_ϕ has been described by Drewnowski and Nawrocki [5].

Denote by τ_ϕ the Mackey topology of (l^ϕ,τ_ϕ), by τ_{h^ϕ} the Mackey topology of $\left(h^\phi,\tau_{\phi|h^\phi}\right)$, and by π_ϕ the topology defined by the Riesz seminorm p_ϕ.

Combining [5, Theorems 5.1 and 5.3] with Lemma 2.2 we get the following important descriptions of τ_{h^ϕ} and τ_ϕ.

THEOREM 2.3. The following equalities hold:

$$\tau_{h^\phi} = \tau_{\bar\phi|h^\phi}, \quad \tau_\phi = (\tau_{\bar\phi|l^\phi}) \vee \pi_\phi .$$

It is well-known (see [11], [12]) that the F-norm topology $\tau_{\bar\phi}$ on $l^{\bar\phi}$ can be generated by two Riesz norms:

$$\|x\|_{\bar\phi} = \inf_{\lambda>0}\left\{\frac{1}{\lambda}(\rho_{\bar\phi}(\lambda x)+1)\right\}$$

$$= \sup\left\{\left|\sum_{i=1}^{\infty} x(i)z(i)\right| : z \in l^{\phi^*}, \rho_{\phi^*}(z) \le 1\right\}$$

and

$$|||\, x \,|||_{\bar\phi} = \inf\{\lambda > 0 : \rho_{\bar\phi}(x/\lambda) \le 1\} .$$

Moreover, $|||\, x \,|||_{\bar\phi} \le \|x\|_{\bar\phi} \le 2\, |||\, x \,|||_{\bar\phi}$ for all $x \in l^{\bar\phi}$ and $|||\, x \,|||_{\bar\phi} \le 1$ iff $\rho_{\bar\phi}(x) \le 1$.

Therefore, in view of Theorem 2.3 the Mackey topology τ_ϕ can be generated by two Riesz norms:

$$p_\phi \vee \|\cdot\|_{\bar\phi} \quad \text{and} \quad p_\phi \vee |||\cdot|||_{\bar\phi}$$

which will be of importance in our discussion. Thus two corresponding Riesz norms on $(l^\phi)^*$ can be given by

$$\|f\|_\phi^* = \sup\left\{|f(x)| : x \in l^\phi,\ \ p_\phi(x) \le 1 \ \text{ and } \ |||\, x \,|||_{\bar\phi} \le 1\right\}$$

$$|||\, f \,|||_\phi^* = \sup\left\{|f(x)| : x \in l^\phi,\ \ p_\phi(x) \le 1 \ \text{ and } \ \|x\|_{\bar\phi} \le 1\right\}.$$

Thus $(l^\phi)^*$ is a Banach lattice under each of the norms $\|\cdot\|_\phi^*$ and $|||\cdot|||_\phi^*$. Moreover, since $\rho_\phi(x) \le 1$ implies $p_\phi(x) \le 1$ and $\rho_{\bar\phi}(x) \le 1$, we can put (see [19]):

$$\|f\|_{\rho_\phi}^* = \sup\{|f(x)| : x \in l^\phi,\ \ \rho_\phi(x) \le 1\}.$$

We shall denote by $(l^\phi)^\sim$ the collection of all order bounded linear functionals on l^ϕ. It is well-known that $(l^\phi)^\sim = (l^\phi)^*$ (see [1, Theorem 16.9]). An order bounded linear functional f on l^ϕ is said to be <u>order continuous</u> (resp. <u>singular</u>) if $x_\alpha \xrightarrow{0} 0$ in l^ϕ implies $f(x_\alpha) \to 0$ for a net (x_α) in l^ϕ (resp. $f(x) = 0$ for all $x \in h^\phi$) (see [9, Ch. X]). The set of all order continuous (resp. singular) functionals on l^ϕ will be denoted by $(l^\phi)_n^\sim$ (resp. $(l^\phi)_s^\sim$).

The next theorem gives a characterization of the space $(l^\phi)^*$.

THEOREM 2.4. (a) For a linear functional f on l^ϕ the following statements are equivalent:

(1) f is order bounded.

(2) f is τ_ϕ-continuous.

(3) There exist unique $f_n \in (l^\phi)_n^\sim$ and $f_s \in (l^\phi)_s^\sim$ such that

$$f(x) = f_n(x) + f_s(x) \quad \text{for} \quad x \in l^\phi.$$

(b) $(l^\phi)_s^\sim = \left((l^\phi)_n^\sim\right)^d$ (= the disjoint complement of $(l^\phi)_n^\sim$ in $(l^\phi)^*$), and moreover, $(l^\phi)_n^\sim$ and $(l^\phi)_s^\sim$ are Banach lattices under each of the norms $\| \cdot \|_\phi^*$, $||| \cdot |||_\phi^*$.

PROOF. (a) Since $(l^\phi, p_\phi \vee \| \cdot \|_{\bar\phi})^* = (l^\phi)^* = (l^\phi)^\sim$, by [9, Ch. VI, §1, Theorem 5], we obtain that $(l^\phi)_n^\sim$ separates the points of l^ϕ, and to get our result it suffices to use Theorem 6 of [9, Ch. X, §3].

(b) Since $(l^\phi)_n^\sim$ is a band of $(l^\phi)^\sim$ (see [1, Theorem 3.7]) $(l^\phi)_n^\sim$ is a $\| \cdot \|_\phi^*$-closed (resp. $||| \cdot |||_\phi^*$-closed) subspace of $(l^\phi)^*$ (see [1, Theorem 5.6]). Thus $(l^\phi)_n^\sim$ is a Banach lattice, because $(l^\phi)^*$ is a Banach lattice. Moreover, since $(l^\phi)_s^\sim = \left((l^\phi)_n^\sim\right)^d$, $(l^\phi)_s^\sim$ is a band of $(l^\phi)^\sim$ (see [1, p. 27]), and by the above argument $(l^\phi)_s^\sim$ is a Banach lattice.

In view of [17] the <u>conjugate</u> $\bar\rho_\phi$ of the modular ρ_ϕ can be defined on the algebraic dual $\tilde{l}^\phi$ of l^ϕ as follows:

$$\bar\rho_\phi(f) = \sup\{|f(x)| - \rho_\phi(x) : x \in l^\phi\}.$$

Note that if $f \geq 0$, then

$$\bar\rho_\phi(f) = \sup\{f(x) - \rho_\phi(x) : 0 \leq x \in \omega, \rho_\phi(x) < \infty\}.$$

Indeed, since $|f(x)| \leq f(|x|)$ (see [1, p. 21]) and $\rho_\phi(x) = \rho_\phi(|x|)$ we have

$$\bar\rho_\phi(f) \leq \sup f(|x|) - \rho_\phi(|x|) : \rho_\phi(|x|) < \infty\}$$
$$\leq \sup\{f(x) - \rho_\phi(x) : 0 \leq x \in \omega, \rho_\phi(x) < \infty\}.$$

We shall need the following definition.

A linear functional f on l^ϕ is said to be <u>bounded for ρ_ω</u> (see [16], [17]) if there exists $\gamma > 0$ such that

$$|f(x)| \leq \gamma(\rho_\phi(x) + 1) \quad \text{for} \quad x \in l^\phi.$$

The collection of all bounded for ρ_ϕ linear functionals on l^ϕ will be denoted by $\overline{l^\phi}$.

The basic properties of $\bar\rho_\phi$ are included in the following

THEOREM 2.5. The conjugate $\bar\rho_\phi$ of the modular ρ_ϕ is a convex orthogonal additive modular on $\overline{l^\phi}$. Moreover, the following equality holds: $(l^\phi)^* = \overline{l^\phi}$.

Proof. Using [17, §4] and arguing as in the proof of [16, Theorem 38.2] we obtain that $\bar\rho_\phi$ is a convex orthogonal additive modular on $\overline{l^\phi}$. To end the proof it suffices to show that $(l^\phi)^* = \overline{l^\phi}$. Indeed, let $f \in (l^\phi)^*$ and $\rho_\phi(x) < \infty$. Then $p_\phi(x) \leq 1$ and there exists $\gamma > 0$ such that $|f(x)| \leq \gamma\,(\max\,(p_\phi(x), \|x\|_{\bar\phi})) \leq \gamma(\rho_{\bar\phi}(x) + 1)$ $\leq \gamma(\rho_\phi(x) + 1)$, because $\bar\phi(u) \leq \phi(u)$ for $u \geq 0$. Thus $f \in \overline{l^\phi}$; hence $(l^\phi)^* \subset \overline{l^\phi}$. Next, let $f \in \overline{l^\phi}$ and let $|x|_\phi < 1$. Then $\rho_\phi(x) \leq 1$, and hence $|f(x)| \leq 2\gamma$ for some $\gamma > 0$. This means that $f \in (l^\phi)^*$, and thus $\overline{l^\phi} \subset (l^\phi)^*$. The proof is completed.

Thus by means of $\bar\rho_\phi$ two modular norms can be defined on $(l^\phi)^*$ in a usual way (see [16], [17]):

$$\|f\|_{\bar\rho_\phi} = \inf_{\lambda > 0}\left\{\frac{1}{\lambda}(\bar\rho_\phi(\lambda f) + 1)\right\} \quad \text{(the first modular norm)}$$

$$|||f|||_{\bar{\rho}_\phi} = \inf\{\lambda > 0 : \bar{\rho}_\phi(f/\lambda) \le 1\} \quad \text{(the second modular norm).}$$

3. Order Continuous Linear Functionals on l^ϕ. We shall start this section with a description of the Köthe dual $(l^\phi)^\times$ of l^ϕ that will be useful in obtaining a corresponding characterization of order continuous linear functional on l^ϕ (see [20, Proposition 1.9]).

Let us recall that the Köthe dual $S^\times$ of a sequence space S is the sequence space defined by (see [10, §30.1]):

$$S^\times = \left\{ y \in \omega : \sum_{i=1}^{\infty} |x(i)y(i)| < \infty \quad \text{for all} \quad x \in S \right\}.$$

THEOREM 3.1. The following equalities hold:

$$(l^\phi)^\times = (h^\phi)^\times = (h^{\bar{\phi}})^\times = l^{\phi^*}.$$

In particular, if $\liminf_{u \to 0} \phi(u)/u > 0$, then $(l^\phi)^\times = l^\infty$.

PROOF. First, we shall show that $(l^\phi)^\times = (h^\phi)^\times = (h^{\bar{\phi}})^\times$. Since $(l^\phi)^\times \subset (h^\phi)^\times$ and $(h^{\bar{\phi}})^\times \subset (h^\phi)^\times$, it suffices to show that $(h^\phi)^\times \subset (l^\phi)^\times$ and $(h^\phi)^\times \subset (h^{\bar{\phi}})^\times$. Indeed, let $y \in (h^\phi)^\times$, i.e., $\sum_{i=1}^{\infty} |z(i)y(i)| < \infty$ for all $z \in h^\phi$. Putting

$$g_y(z) = \sum_{i=1}^{\infty} z(i)y(i) \quad \text{for} \quad z \in h^\phi,$$

by [20, Proposition 1.9] and Theorem 2.3 we get

$$g_y \in (h^\phi)_n^\sim = (h^\phi)^\sim = \left(h^\phi, \tau_{\phi|h^\phi}\right)^* = \left(h^\phi, \tau_{\bar{\phi}|h^\phi}\right)^*.$$

Therefore, we can put

$$\|g_y\|_{\bar{\phi}} = \sup\left\{ \left| \sum_{i=1}^{\omega} z(i)y(i) \right| : z \in h^\phi, \quad |||z|||_{\bar{\phi}} \le 1 \right\}.$$

Let now $x \in l^\phi$ (resp. $x \in h^{\bar{\phi}}$), $x \neq 0$. We shall show that $\sum_{i=1}^{\infty} |x(i)y(i)| < \infty$. Since $x \in l^{\bar{\phi}}$ and $x^{(n)} \in h^\phi$ we get

$$\frac{1}{|||x|||_{\bar{\phi}}} \sum_{i=1}^{\infty} |x(i)y(i)| = \frac{1}{|||x|||_{\bar{\phi}}} \sup_n \sum_{i=1}^{\infty} |x^{(n)}(i)| \cdot \operatorname{sign} y(i) \cdot y(i)$$

$$\le \sup\left\{ \left| \sum_{i=1}^{\infty} z(i)y(i) \right| : z \in h^\phi, \quad |||z|||_{\bar{\phi}} \le 1 \right\} = \|g_y\|_{\bar{\phi}} < \infty.$$

Hence $y \in (l^\phi)^\times$ (resp. $y \in (h^{\bar{\phi}})^\times$), so that $(l^\phi)^\times = (h^\phi)^\times = (h^{\bar{\phi}})^\times$.

We have $(h^{\bar{\phi}})_n^\sim = (h^{\bar{\phi}})^\sim = \left(h^{\bar{\phi}}, \tau_{\bar{\phi}|h^{\bar{\phi}}}\right)^*$. It is well-known that by the mapping $(y \to g_y)$ the space $(h^{\bar{\phi}})^\times$ can be identified with $(h^{\bar{\phi}})_n^\sim$ (see [20, Proposition 1.9]), and the space $l^{\bar{\phi}^*}$ with $\left(h^{\bar{\phi}}, \tau_{\bar{\phi}|h^{\bar{\phi}}}\right)$ (see [12, Ch. II, §3, Theorem 2]). Thus $(h^{\bar{\phi}})^\times = l^{\bar{\phi}^*}$, and since $\bar{\phi}^* = \phi^{***} = \phi^*$, the proof is complete.

REMARK. The equality $(l^\phi)^\times = l^{\phi^*}$ has been obtained by the author in [18] in a different way, using the so-called modular topology on l^ϕ.

REMARK. Assume now that ϕ is an Orlicz function, not necessarily satisfying the condition: $\phi(u)/u \to \infty$ as $u \to \infty$. Let ψ be any Orlicz function such that $\psi(u) = \phi(u)$ for $0 \le u \le 1$, and $\psi(u)/u \to \infty$ as $u \to \infty$. Then in view of Theorem 3.1 we get $(l^\phi)^\times = (l^\psi)^\times = l^{\psi^*}$. Thus, by Lemma 3.1 we get $(l^p)^\times = l^\infty$ for $0 < p \le 1$.

We are now able to give a characterization of order continuous linear functionals on l^ϕ.

THEOREM 3.2. Let f be a linear functional on l^ϕ.

(a) The following statements are equivalent:

(1) f is order continuous.

(2) There exists a unique $y \in 1^{\phi^*}$ such

$$f(x) = f_y(x) = \sum_{i=1}^{\infty} x(i)y(i) \qquad \text{for all } x \in 1^{\phi}.$$

(b) If f is order continuous, then the following equalities hold:

$$\bar{\rho}_{\phi}(f) = \rho_{\phi^*}(y),$$

$$\|f\|_{\phi}^{*} = \|f\|_{\bar{\rho}_{\phi}} = \|y\|_{\phi^*},$$

$$|||f|||_{\phi}^{*} = |||f|||_{\bar{\rho}_{\phi}} = |||y|||_{\phi^*}.$$

(c) Moreover, the map $1^{\phi^*} \supset y \to f_y \in (1^{\phi})_n^{\sim}$ is a Riesz isomorphism.

PROOF. (a) It follows from [20, Proposition 1.9] and Theorem 3.1.

(b) By (a) we have $f(x) = \sum_{i=1}^{\infty} x(i)y(i)$ for some $y \in 1^{\phi^*}$ and all $x \in 1^{\phi}$.

First, we shall show that $\bar{\rho}_{\phi}(f) = \rho_{\phi^*}(y)$. From the definition of ϕ^* we easily obtain that $\bar{\rho}_{\phi}(f) \le \rho_{\phi^*}(y)$. To prove that $\bar{\rho}_{\phi}(f) \ge \rho_{\phi^*}(y)$ let us note that there exists $0 \le z \in \omega$ such that

$$\phi(z(i)) + \phi^*(|y(i)|) = |z(i)y(i)| \quad \text{for} \quad i = 1,2,\ldots.$$

Putting $x(i) = (\operatorname{sign} y(i)) \cdot z(i)$ for $i = 1,2,\ldots$, we get

$$\begin{aligned}\rho_{\phi^*}(y) &= \sum_{i=1}^{\infty} \phi^*(|y(i)|) \\ &= \sup_n \left\{ \sum_{i=1}^{n} |z(i)y(i)| - \sum_{i=1}^{n} \phi(z(i)) \right\} \\ &\le \sup_n \left\{ \left| \sum_{i=1}^{\infty} x^{(n)}(i)y(i) \right| - \sum_{i=1}^{\infty} \phi(|x^{(n)}(i)|) \right\} \le \bar{\rho}_{\phi}(f).\end{aligned}$$

In turn, we shall show that $\|f\|_{\phi}^{*} = \|y\|_{\phi^*}$. We have $\|y\|_{\phi^*} = \sup\left\{ \left| \sum_{i=1}^{\infty} z(i)y(i) \right| : x \in 1^{\bar{\phi}},\ \rho_{\bar{\phi}}(z) \le 1 \right\}$, and hence $\|f\|_{\phi}^{*} \le \|y\|_{\phi^*}$. On the other hand, let $z \in 1^{\bar{\phi}}$ with $\rho_{\bar{\phi}}(z) \le 1$. Putting $x(i) = (\operatorname{sign} y(i)) \cdot |z(i)|$ $(i = 1,2,\ldots)$, we have $p_{\phi}(x^{(n)}) = 0$ and $\rho_{\bar{\phi}}(x^{(n)}) \le \rho_{\bar{\phi}}(z) \le 1$. Thus

$$\begin{aligned}\left| \sum_{i=1}^{\infty} z(i)y(i) \right| &\le \sup_n \sum_{i=1}^{\infty} |z^{(n)}(i)y(i)| \\ &= \sup_n \left| \sum_{i=1}^{\infty} x^{(n)}(i)y(i) \right| \le \|f\|_{\phi}^{*}.\end{aligned}$$

Thus $\|y\|_{\phi^*} \le \|f\|_{\phi}^{*}$, and hence $\|f\|_{\phi}^{*} = \|y\|_{\phi^*}$.

Moreover, since $\bar{\rho}_{\phi}(\lambda f) = \rho_{\phi^*}(\lambda y)$ for $\lambda > 0$, we get $\|f\|_{\bar{\rho}_{\phi}} = \|y\|_{\phi^*}$.

Next, we shall show that $|||f|||_{\phi}^{*} \le |||y|||_{\phi^*}$. To prove that $|||f|||_{\phi}^{*} \le |||y|||_{\phi^*}$, let us assume that $x \in 1^{\phi}$, $p_{\phi}(x) \le 1$ and $\|x\|_{\bar{\phi}} \le 1$. Then $x \in 1^{\bar{\phi}}$, and by the Hölder's inequality (see [11,§9]) we get $|f(x)| \le \|x\|_{\bar{\phi}^*} \cdot |||y|||_{\phi^*} \le |||y|||_{\phi^*}$, because $\bar{\phi}^* = \bar{\phi}$. Thus $|||f|||_{\phi}^{*} \le |||y|||_{\phi^*}$. To prove that $|||y|||_{\phi^*} \le |||f|||_{\phi}^{*}$ let us note that (see [11, p. 135]):

$$|||y|||_{\phi^*} = \sup\left\{\left|\sum_{i=1}^{\infty} z(i)y(i)\right| : z \in l^{\bar{\phi}},\quad \|z\|_{\bar{\phi}} \le 1\right\}.$$

Let now $z \in l^{\bar{\phi}}$ and $\|z\|_{\bar{\phi}} \le 1$. Putting $x(i) = (\operatorname{sign} y(i)) \cdot |z(i)|$ $(I = 1,2,\ldots)$ we have $p_\phi(x^{(n)}) = 0$, $\|x(n)\|_{\bar{\phi}} \le \|z\|_{\bar{\phi}} \le 1$, and as above we get $|||y|||_{\phi^*} \le |||f|||^*_\phi$.

Finally, since $\bar{\rho}_\phi(f/\lambda) = \rho_{\phi^*}(y/\lambda)$ for $\lambda > 0$, we get $|||f|||_{\bar{\rho}_\phi} = |||y|||_{\phi^*}$.

(c) See [9, Ch. VI, §1, Theorem 1] and [14, Theorem 18.5].

REMARK. The general form of ϕ-continuous (continuous with respect to the modular ρ_ϕ) linear functionals on an Orlicz space $L^\phi(a,b)$ defined by an Orlicz function satisfying conditions $\phi(u)/u \to 0$ as $u \to 0$ and $\phi(u)/u \to \infty$ as $u \to \infty$, has been found by W. Orlicz [19].

4. Singular Linear Functionals on l^ϕ. In this section we assume that ϕ does not satisfy the Δ_2-condition at 0, because otherwise $(l^\phi)^\sim_s = \{0\}$.

The following lemma describes positive singular linear functionals on l^ϕ.

LEMMA 4.1. Let f be a positive singular linear functional on l^ϕ.

(a) For any $\varepsilon > 0$ there exists $0 \le y \in \omega$ with $\rho_\phi(y) < \varepsilon$ such that $\|f\|^*_\phi \le f(y)$.

(b) The following equalities hold:

$$\rho_{\bar{\phi}}(f) = \|f\|^*_{\rho_\phi} = \|f\|^*_\phi = |||f|||^*_\phi$$
$$= \sup\{f(x) : 0 \le x \in \omega,\quad \rho_\phi(x) < \infty\}.$$

(c) There exists $0 \le y \in \omega$ with $\rho_\phi(y) < \infty$ such that

$$\|f_A\|^*_\phi = f(y_A) \text{ for any subset } A \text{ of } \mathbf{N}$$

and

$$p_\phi(y_A) = 1 \text{ for any subset } A \text{ of } \mathbf{N} \text{ with } \|f_A\|^*_\phi \ne 0.$$

PROOF. (a) Let $\varepsilon > 0$ be given. Since (see [26, Lemma 102.1])

$$\|f\|^*_\phi = \sup\{f(x) : 0 \le x \in l^\phi,\quad p_\phi(x) \le 1,\quad \rho_{\bar{\phi}}(x) \le 1\},$$

for every $k \in N$ there exists $0 \le z_k \in l^\phi$ such that $p_\phi(z_k) < 1$ and $\|f\|^*_\phi \le f(z_k) + \frac{1}{k}$. Then $\rho_\phi(z_k) < \infty$ and there exists a strictly increasing sequence of natural numbers (n_k) such that

$$\rho_\phi\left(z_k - z_k^{(n_k)}\right) = \sum_{i=n_k}^{\infty} \phi(z_k(i)) < \frac{\varepsilon}{2^k}.$$

Let $x_k = z_k - z_k^{(n_k)}$ for $k = 1,2,\ldots$. Then in view of the axion (C) of completeness of the modular ρ_ϕ there exists $0 \le y \in \omega$ such that $x_k \le y$, for all $k \in N$, and $\rho_\phi(y) \le \sum_{k=1}^{\infty} \rho_\phi(x_k) < \varepsilon$. But $z_k^{(n_k)} \in h^\phi$ for all $k \in \mathbf{N}$, so that

$$\|f\|^*_\phi \le f\left(z_k - z_k^{(n_k)}\right) + f\left(z_k^{(n_k)}\right) + \frac{1}{k}$$
$$= f(x_k) + \frac{1}{k} \le f(y) + \frac{1}{k}.$$

Since $\varepsilon > 0$ and k are arbitrary, we conclude that $\|f\|^*_\phi \le f(y)$.

(b) We have

$$|||f|||^*_\phi \le \|f\|^*_\phi \le \sup\left\{f(x) : 0 \le x \in l^\phi,\quad p_\phi(x) \le 1,\quad \rho_{\bar{\phi}}(x) < \infty\right\}.$$

To prove that $\sup\{f(x) : 0 \le x \in l^\phi,\ p_\phi(x) \le 1,\ \rho_{\bar{\phi}}(x) < \infty \le |||f|||^*_\phi$ assume that $0 \le x \in l^\phi$ and

$p_\phi(x) \le 1$, $\rho_{\bar\phi}(x) < \infty$. Given an $\eta > 0$, there exists $n \in \mathbf{N}$ such that $\rho_{\bar\phi}(x - x^{(n)}) < \eta$. Then

$$\|x - x^{(n)}\|_{\bar\phi} \le 1 + \rho_\phi(x - x^{(n)}) \le 1 + \eta$$

and

$$f(x) = f(x - x^{(n)}) + f(x^{(n)}) = f(x - x^{(n)})$$
$$\le (1 + \eta)\, |||f|||_\phi^\bullet .$$

Hence $f(x) \le |||f|||_\phi^\bullet$, and thus we obtain

$$|||f||| = \|f\|_\phi^\bullet = \sup\left\{ f(x) : x \in 1^\phi,\ \ p_\phi(x) \le 1,\ \ \rho_{\bar\phi}(x) < \infty \right\}.$$

Moreover, by (a) there exists $0 \le y \in \omega$, with $\rho_\phi(y) \le 1$, such that $\|f\|_\phi^\bullet \le f(y)$. Hence

$$\|f\|_{\rho_\phi}^\bullet = \sup\{f(x) : 0 \le x \in \omega,\ \ \rho_\phi(x) \le 1\}$$
$$\le \sup\{f(x) : 0 \le x \in \omega,\ \ \rho_\phi(x) < \infty\}$$
$$\le \sup\left\{ f(x) : x \in 1^\phi,\ \ p_\phi(x) \le 1,\ \ \rho_{\bar\phi}(x) < \infty \right\}$$
$$= \|f\|_\phi^\bullet \le f(y) \le \sup\{f(x) : 0 \le x \in \omega,\ \ \rho_\phi(x) \le 1\}.$$

Thus we proved that

$$\|f\|_{\rho_\phi}^\bullet = |||f|||_\phi^\bullet = \|f\|_\phi^\bullet = \sup\{f(x) : 0 \le x \in \omega, \rho_\phi(x) < \infty\}.$$

Finally, we shall show that $\bar\rho_\phi(f) = \|f\|_\phi^\bullet$. Indeed, by (a), for every $n \in \mathbf{N}$, there exists $0 \le y_n \in \omega$, with $\rho_\phi(y_n) \le \frac{1}{n}$, and such that $\|f\|_\phi^\bullet \le f(y_n)$. Hence

$$\bar\rho_\phi(f) = \sup\{f(x) - \rho_\phi(x) : 0 \le x \in \omega,\ \ \rho_\phi(x) < \infty\}$$
$$\ge f(y_n) - \rho_\phi(y_n) \ge \|f\|_\phi^\bullet - \frac{1}{n}.$$

Hence $\bar\rho_\phi(f) \ge \|f\|_\phi^\bullet$, and since

$$\bar\rho_\phi(f) \le \sup\{f(x) : 0 \le x \in \omega,\ \ \rho_\phi(x) < \infty\} = \|f\|_\phi^\bullet$$

we get $\bar\rho_\phi(f) = \|f\|_\phi^\bullet$. Thus the proof of (b) is completed.

(c) Let A be a subset of N, and let $0 \le x \in \omega$ with $\rho_\phi(x) < \infty$ be given. Arguing as in (a) we obtain that there exists $0 \le z_k \in \omega$ with $\rho_\phi(z_k) < \infty (k = 1, 2, \ldots)$ such that $\|f\|_\phi^\bullet \le f(z_k) + \frac{1}{k}$. Since $\|f\|_\phi^\bullet = \sup\{f(z) : 0 \le z \in \omega, \rho_\phi(z) < \infty\}$ (see (b)), we have

$$f(x \vee z_k) \le f(z_k) + \frac{1}{k}.$$

for all $k \in N$, because $\rho_\phi(x \vee z_k) \le \rho_\phi(x) + \rho_\phi(z_k) < \infty$. But $(x \vee z_k - z_k)_A \le x \vee z_k - z_k$, so we get

$$f(x_A) \le f((x \vee z_k)_A) \le f((z_k)_A) + \frac{1}{k} \quad (k = 1, 2, \ldots).$$

Choose an increasing sequence of natural numbers (m_k) such that $\rho_\phi\left(z_k - z^{(m_k)}\right) < \frac{1}{2^k}$, and let $x_k = z_k - z_k^{(m_k)}$. Then in view of the axiom (C) of completeness of ρ_ϕ, there exists $0 \le y \in \omega$ such that $x_k \le y$ for all $k \in \mathbf{N}$, and $\rho_\phi(y) \le 1$. Hence

$$f(x_A) \le f\left(\left(z_k - z_k^{(m_k)}\right)_A\right) + f\left(\left(z_k^{(m_k)}\right)_A\right) + \frac{1}{k}$$
$$= f((x_k)_A) + \frac{1}{k} \le f(y_A) + \frac{1}{k}.$$

Thus we obtain that $\|f_A\|_\phi^* = f(y_A)$, because by (b),

$$\|f_A\|_\phi^* = \sup\{f(x_A) : 0 \le x \in \omega, \quad \rho_\phi(x) < \infty\}.$$

Assume now that $\|f_A\|_\phi^* \neq 0$. Given $\eta > 0$ we have $\rho_\phi(y_A/(p_\phi(y_A)+\eta)) < \infty$, and hence, by (b), $\|f_A\|_\phi^* \ge f((y_A/(p_\phi(y_A)+\eta))$. Thus $\|f_A\|_\phi^* = f(y_A) \le (p_\phi(y_A)+\eta)\|f_A\|_\phi^*$, so $p_\phi(y_A) = 1$, because $p_\phi(y_A) \le p_\phi(y) \le 1$. Thus the proof of (c) is completed.

COROLLARY 4.2. The space $\left((1^\phi)_s^\sim, \|\cdot\|_\phi^*\right)$ is an abstract L-space.

PROOF. By Theorem 2.4, $\left((1^\phi)_s^\sim, \|\cdot\|_\phi^*\right)$ is a Banach lattice. Arguing as in the proof of Lemma 2 of [2] we can show that $\|f_1+f_2\|_\phi^* = \|f_1\|_\phi^* + \|f_2\|_\phi^*$ for any $f_1, f_2 \in \left((1^\phi)_s^\sim\right)^+$, and this means that $(1^\phi)_s^\sim$ is an abstract L-space (see [23, Ch. II, §9]).

By $ba(\mathrm{N})$ we denote the family of all bounded real valued finitely additive set functions on N. It is known that $ba(\mathrm{N})$ is a vector lattice with the usual ordering: $\nu_1 \ge \nu_2$ iff $\nu_1(A) \ge \nu_2(A)$ for all $A \subset \mathrm{N}$. Then $\nu = \nu^+ - \nu^-$ and $|\nu| = \nu^+ + \nu^-$, where ν^+ and ν^- denote the positive and the negative part of $\nu \in ba(\mathrm{N})$. Moreover $ba(\mathrm{N})$ is a Banach space under the norm $\|\nu\| = |\nu|(\mathrm{N})$ (see [6, Ch. III, 1.4, 1.7]).

For given $f \in \left((1^\phi)_s^\sim\right)^+$ let us put $\nu_f(A) = \|f_A\|_\phi^*$ for any subset A of N. Then by Corollary 4.2, $\nu_f \in (ba(\mathrm{N}))^+$ and $\|\nu_f\| = \nu_f(\mathrm{N}) = \|f\|_\phi^*$.

The following definition is justified by Lemma 4.1.

A $\nu \in ba(\mathrm{N})$ is said to be in class $B_\phi(\mathrm{N})$ if there exists $0 \le y \in \omega$, with $\rho_\phi(y) < \infty$, such that $p_\phi(y_A) = 1$ for any subset A of N with $|\nu|(A) \neq 0$.

One can show that $B_\phi(\mathrm{N})$ is a Riesz subspace of $ba(\mathrm{N})$. In view of Lemma 4.1 we have the following

LEMMA 4.3. If $f \in \left((1^\phi)_s^\sim\right)^+$, then $\nu_f \in (B_\phi(\mathrm{N}))^+$.

Thus we can define a mapping $T : \left((1^\phi)_s^\sim\right)^+ \to (B_\phi(\mathrm{N}))^+$ given by

$$T(f) = \nu_f \quad \text{for any} \quad f \in \left((1^\phi)_s^\sim\right)^+.$$

In view of Corollary 4.2 the mapping T is additive.

For any $\nu \in (ba(\mathrm{N}))^+$ we define a positive functional I_ν on $(1^\phi)^+$ by

$$I_\nu(x) = \inf\left\{\sum_{k=1}^{n} p_\phi(x_{A_k})\nu(A_k)\right\}$$

where the infimum is taken over all finite disjoint partitions $(A_k)_1^n$ of N.

By the same argument as in the proof of Lemma 5 of [2] we can prove that the functional I_ν is additive on $(1^\phi)^+$. Thus I_ν has a unique positive extension to a linear functional on 1^ϕ (see [1, Lemma 3.1]). This extension (denoted again by I_ν) is given by $I_\nu(x) = I_\nu(x^+) - I_\nu(x^-)$ for all $x \in 1^\phi$.

LEMMA 4.4. If $\nu \in (ba(\mathrm{N}))^+$, then $I_\nu \in \left((1^\phi)_s^\sim\right)^+$ and $\|I_\nu\|_\phi^* \le \nu(\mathrm{N})$.

PROOF. Since I_ν is positive on l^ϕ, I_ν is order bounded. It is seen that $I_\nu(x) = 0$ for all $x \in h^\phi$, so $I_\nu \in \left((1^\phi)_s^\sim\right)^+$. Moreover, $|I_\nu(x)| \le I_\nu(x^+) + I_\nu(x^-) = I_\nu(|x|) \le p_\phi(x)\nu(\mathrm{N})$ for all $x \in 1^\phi$, so $\|I_\nu\|_\phi^* \le \nu(\mathrm{N})$.

Thus we can define a mapping $G : (B_\phi(\mathrm{N}))^+ \to \left((1^\phi)^{\sim}_s\right)^+$ by

$$G(\nu) = I_\nu \quad \text{for any} \quad \nu \in (B_\phi(\mathrm{N}))^+.$$

THEOREM 4.5. The following statements hold:

(1) $(G \circ T)(f) = f$ for any $f \in \left((1^\phi)^{\sim}_s\right)^+$, i.e.,

$$f(x) = I_{\nu_f}(x) \quad \text{for all} \quad x \in 1^\phi.$$

(2) $(T \circ G)(\nu) = \nu$ for any $\nu \in (B(\mathrm{N}))^+$, i.e.,

$$\nu(A) = \|(I_\nu)_A\|^*_\phi \quad \text{for any subset } A \text{ of } \mathrm{N}.$$

PROOF. (1) Using Corollary 4.2 and Lemma 4.4, it suffices to repeat the arguments of the proof of Theorem 2 of [2].

(2) We first prove the case $A = \mathrm{N}$. Since $\nu \in (B_\phi(\mathrm{N}))^+$, there exists $0 \le y \in \omega$ such that $\rho_\phi(y) < \infty$ and $p_\phi(y_E) = 1$ for any subset E of N with $\nu(E) > 0$. Then for any finite disjoint partition $(E_k)_1^n$ of N we have $\sum_{k=1}^{n} p_\phi(y_{E_k})\nu(E_k) = \nu(\mathrm{N})$, so $I_\nu(y) = \nu(\mathrm{N})$. According to Lemma 4.1, we have $\|I_\nu\|^*_\phi \ge I_\nu(y) = \nu(\mathrm{N})$. Moreover, we have $I_\nu(x) \le p_\phi(x)\nu(\mathrm{N})$ for all $0 \le x \in 1^\phi$. Hence $\|I_\nu\|^*_\phi \le \nu(\mathrm{N})$, so $\|I_\nu\|^*_\phi = \nu(\mathrm{N})$. Assume now that A is a fixed subset of N, and let $\nu_1(B) = \nu(A \cap B)$ for any $B \subset \mathrm{N}$. One can easily show that $I_{\nu_1} = (I_\nu)_A$. Hence, by the above, we get $\|(I_\nu)_A\|^*_\phi = \|I_{\nu_1}\|^*_\phi = \nu_1(\mathrm{N}) = \nu(\Lambda)$, and the proof is completed.

By Theorem 4.5 the mapping G is additive, because T is additive. Thus T and G have unique positive extensions to linear mappings $\tilde{T} : (1^\phi)^{\sim}_s \to B_\phi(\mathrm{N})$ and $\tilde{G} : B_\phi(\mathrm{N}) \to (1^\phi)^{\sim}_s$ (see [1, Lemma 3.1]) given by

$$\tilde{T}(f) = \nu_{f^+} - \nu_{f^-} \quad \text{and} \quad \tilde{G}(\nu) = I_{\nu^+} - I_{\nu^-}$$

Let us put: $\nu_f = \nu_{f^+} - \nu_{f^-}$ and $I_\nu = I_{\nu^+} - I_{\nu^-}$. For any $\nu \in B_\phi(\mathrm{N})$ we shall write

$$\int x d\nu = I_\nu(x) \quad \text{for all} \quad x \in 1^\phi.$$

THEOREM 4.6. (see [2, Theorem 4]). The mapping $\tilde{T} : (1^\phi)^{\sim}_s \to B_\phi(\mathrm{N})$ is a Riesz isomorphism.

PROOF. In view of Theorem 4.5, we get $(\tilde{G} \circ \tilde{T})(f) = f$, for any $f \in (1^\phi)^{\sim}_s$, and $(\tilde{T} \circ \tilde{G})(\nu) = \nu$, for any $\nu \in B_\phi(\mathrm{N})$. Thus $\tilde{T}$ is a Riesz isomorphism, because $\tilde{T}$ is positive (see [14, Theorem 18.5]).

The final result of this section gives a characterization of singular linear functionals on 1^ϕ.

THEOREM 4.7. Let f be a linear functional on 1^ϕ.

(a) The following statements are equivalent:

(1) f is singular.

(2) There exists a unique $\nu \in B_\phi(\mathrm{N})$ such that

$$f(x) = \int x d\nu \quad \text{for all} \quad x \varepsilon 1^\phi.$$

(b) If f is singular, then the following equalities hold:

$$\bar{\rho}_\phi(f) = \|f\|^*_{\rho_\phi} = \|f\|^*_\phi = |||f|||^*_\phi = \|f\|_{\bar{\rho}_\phi} = |||f|||_{\bar{\rho}_\phi} = |\nu|(\mathrm{N}).$$

PROOF. (a) See the proof of Theorem 4.6.

(b) According to Theorem 4.6, we get $\nu_{|f|}(\mathrm{N}) = |\nu_f|(\mathrm{N})$. Thus, in view of Lemma 4.1, we get

$$\bar{\rho}_\phi(f) = \bar{\rho}_\phi(|f|) = \| \, |f| \, \|^*_{\rho_\phi} = \| \, |f| \, \|^*_\phi = ||| \, |f| \, |||^*_\phi = |\nu_f|(\mathrm{N}).$$

Moreover, since $\bar{\rho}_\phi(\lambda f) = \bar{\rho}_\phi(\lambda |f|) = \lambda \bar{\rho}_\phi(f)$ for $\lambda > 0$ (see Lemma 4.1), we obtain that $\|f\|_{\bar{\rho}_\phi} = \bar{\rho}_\phi(f)$ and $||| f |||_{\bar{\rho}_\phi} = \bar{\rho}_\phi(f)$. Since the norms which occur in our theorem are Riesz norms the proof is complete.

Since $\left((1^\phi)^\sim_s, \|\cdot\|^*_\phi\right)$ is an abstract L-space (see Corollary 4.2), by Theorems 4.6 and 4.7, we obtain that $B_\phi(\mathrm{N})$ is also an abstract L-space.

5. The General Form of Continuous Linear Functionals on 1^ϕ. We are now in position to give a desired characterization of the dual space $(1^\phi)^*$.

THEOREM 5.1. Let f be a linear functional on 1^ϕ.

(a) The following statements are equivalent:

(1) f is τ_ϕ-continuous.

(2) f is order bounded.

(3) There exist unique $y \in 1^{\phi^*}$ and $\nu \in B_\phi(\mathrm{N})$ such that

$$f(x) = \sum_{i=1}^{\infty} x(i)y(i) + \int x d\nu \quad \text{for all} \quad x \in 1^\phi.$$

(b) If f is τ_ϕ-continuous, then the following equalities hold:

$$\bar{\rho}_\phi(f) = \rho_{\phi^*}(y) + |\nu|(\mathrm{N}),$$
$$\|f\|^*_\phi = \|f\|_{\bar{\rho}_\phi} = \|y\|_{\phi^*} + |\nu|(\mathrm{N}).$$

(c) The space h^ϕ is an M-ideal of $(1^\phi, p_\phi \vee ||| \cdot |||_\phi)$.

PROOF. (a) It follows from Theorem 2.4, Theorem 3.2 and Theorem 4.7.

(b) By Theorem 2.4, we have $f = f_n + f_s$, and it is known that $|f|_n = |f_n|$, $|f|_s = |f_s|$, and $|f_n| \wedge |f_s| = 0$. Since the conjugate modular $\bar{\rho}_\phi$ is orthogonal additive on $(1^\phi)^*$, by Theorem 3.2 and Theorem 4.7, we get $\bar{\rho}_\phi(f) = \bar{\rho}_\phi(f_n) + \bar{\rho}_\phi(f_s) = \rho_{\phi^*}(y) + |\nu|(\mathrm{N})$.

We shall now show that $\|f\|^*_\phi = \|y\|_{\phi^*} + |\nu|(\mathrm{N})$. Indeed, let $\varepsilon > 0$ be given. Then there exists $0 \le x \in 1^\phi$ with $p_\phi(x) < 1$, $\rho_{\bar{\phi}}(x) < 1$, such that

$$\|f_n\|^*_\phi = \| \, |f|_n \, \|^*_\phi \le |f|_n(x) + \varepsilon.$$

Moreover, in view of Lemma 4.1 there exists $0 \le y \in \omega$ with $\rho_\phi(y) \le 1 - \rho_{\bar{\phi}}(x)$ such that

$$\|f_s\|^*_\phi = \| \, |f|_s \, \|^*_\phi \le |f|_s(y).$$

Let $z = x \vee y$. Then $\rho_{\bar{\phi}}(z) \le \rho_{\bar{\phi}}(x) + \rho_{\bar{\phi}}(y) \le 1$. Moreover, since $p_\phi(x) < 1$, we have $\rho_\phi(x) < \infty$. Hence $\rho_\phi(z) < \infty$, so $p_\phi(z) \le 1$. Thus

$$\begin{aligned} \|f_n\|^*_\phi + \|f_s\|^*_\phi &\le |f|_n(x) + |f|_s(y) + \varepsilon \\ &\le |f|_n(z) + |f|_s(z) + \varepsilon \\ &= |f|(z) + \varepsilon \le \|f\|^*_\phi + \varepsilon. \end{aligned}$$

Hence $\|f_n\|_\phi^* + \|f_s\|_\phi^* = \|f\|_\phi^*$, and, according to Theorem 3.2 and Theorem 4.7, we obtain $\|f\|_\phi^* = \|y\|_{\phi^*} + |\nu|(\mathrm{N})$. Finally, since $\bar{\rho}_\phi(\lambda f_n) = \rho_{\phi^*}(\lambda y)$ and $\bar{\rho}_\phi(\lambda f_s) = \lambda|\nu|(\mathrm{N})$ for $\lambda > 0$, we easily obtain that $\|f\|_{\bar{\rho}_\phi} = \|y\|_{\phi^*} + |\nu|(\mathrm{N})$.

(c) It is well known that $(h^\phi)^0 = (1^\phi)_s^\sim$ (see [26, Theorem 88.10]), where $(h^\phi)^0$ denotes the annihilator of h^ϕ in $(1^\phi)^*$. Therefore, from (b) it follows that $(h^\phi)^0$ is an L-summand of $\left((1^\phi)^*, \|\cdot\|_\phi^*\right)$ (see [3, Definition 1.1]). According to [3, Definition 2.1] it means that h^ϕ is an M-ideal of $(1^\phi, p_\phi \vee ||| \cdot |||_{\bar{\phi}})$.

REMARK. For a convex Orlicz function ϕ the equality $\|f\|_\phi^* = \|f\|_{\bar{\rho}_\phi}$ has been proved by W. A. Luxemburg and A. C. Zaanen [12, Theorem 5].

As an application of Theorem 5.1 we obtain that continuous linear functionals on h^ϕ have the unique norm preserving extension to 1^ϕ.

COROLLARY 5.3. (see [21, Proposition 3]). Let g be a $\tau_{\phi|h^\phi}$-continuous linear functional on h^ϕ. Then there exists a unique τ_ϕ-continuous linear functional f on 1^ϕ such that $f(x) = g(x)$ for all $x \varepsilon h^\phi$, and $\|g\|_{h^\phi}^* = \|f\|_\phi^*$, where

$$\|g\|_{h^\phi}^* = \sup\left\{|g(x)| : x \in h^\phi, |||x|||_{\bar{\phi}} \le 1\right\} .$$

PROOF. Since $\left(h^\phi, \tau_{\phi|h^\phi}\right)^* = (h^\phi)^\sim = (h^\phi)_n^\sim$ (see [1, Theorem 16.9]), according to [20, Proposition 1.9] and Theorem 3.1 there exists a unique $y \in 1^{\phi^*}$ such that $g(x) = \sum_{i=1}^{\infty} x(i)y(i)$ for all $x \in h^\phi$. Let us put

$$f(x) = \sum_{i=1}^{\infty} x(i)y(i) \quad \text{for all} \quad x \in 1^\phi .$$

Then $f(x) = g(x)$ for $x \in h^\phi$, and, according to Theorem 3.2, f is order continuous and $\|f\|_\phi^* = \|y\|_{\phi^*}$. Now we shall show that $\|g\|_{h^\phi}^* = \|f\|_\phi^*$. Indeed, we have $\|g\|_{h^\phi}^* \le \|f\|_\phi^*$. Let $x \in 1^\phi$ with $p_\phi(x) \le 1$, $|||x|||_{\bar{\phi}} \le 1$. Then

$$\left|\sum_{i=1}^{\infty} x(i)y(i)\right| \le \sup_n \sum_{i=1}^{n} |x(i)y(i)|$$

$$= \sup_n \sum_{i=1}^{\infty} |x^{(n)}(i)| \cdot \operatorname{sign} y(i) \cdot y(i) \le \|g\|_{h^\phi}^* .$$

Hence $\|f\|_\phi^* \le \|g\|_{h^\phi}^*$, and we are done.

Now assume that $\bar{f}$ is another such extension of g, and let $F = \bar{f} - f$. Then F is singular on 1^ϕ and $\bar{f} = f + F$. Hence, by Theorem 2.4, we have $f = \bar{f}_n$ and $F = \bar{f}_s$. Therefore, in view of Theorem 5.1, we have $\|\bar{f}\|_\phi^* = \|f\|_\phi^* + \|F\|_\phi^* = \|y\|_{\phi^*} + \|F\|_\phi^*$. Since $\|\bar{f}\|_\phi^* = \|g\|_{h^\phi}^* = \|y\|_{\phi^*}$, we obtain that $F = 0$, so $\bar{f} = f$. Thus the proof is completed.

REFERENCES

[1] ALIPRANTIS, C. D. and BURKINSHAW, O. Locally Solid Riesz Spaces, Academic Press, New York (1978).

[2] ANDO, T. Linear Functionals on Orlicz Spaces, Niew Arch. Wisk. 8 (1960), 1-16.

[3] BEHRENDS, E. M-Strukture and the Banach-Stone Theorem, Springer-Verlag, Lecture Notes in Math. 736, Berlin, Heidelberg, New York, 1979.

[4] BIRNBAUM, Z. and ORLICZ, W. Über die verallgemeinerung des begriffes der zueinander potenzen, Studia Math. 3 (1931), 1-67.

[5] DREWNOWSKI, L. and NAWROCKI, M. On the Mackey Topology of Orlicz Sequence Spaces, Arch. Math. 37 (1981), 256-266.

[6] DUNFORD, N. and SCHWARTZ, J. T. Linear Operators, Part I: General Theory, Interscience, New York, 1958.

[7] FERNANDEZ, R. Characterization of the Dual of an Orlicz Space, Comment. Math. (to appear).

[8] KALTON, N. J. Orlicz Sequence Spaces Without Local Convexity, Math. Proc. Camb. Phil. Soc. 81 (1977), 253-277.

[9] KANTOROVICH, L. V. and AKILOV, G. P. Functional Analysis, Moscow, 1984 (Russian).

[10] KÖTHE, G. Topological Vector Spaces I, Springer, Berlin, Heidelberg, New York, 1983.

[11] KRASNOSELSKII, M. and RUTICHII, YA. B. Convex Functions and Orlicz Spaces, P. Nordhoff Ltd., Groningen, 1961.

[12] LUXEMBURG, W. A. Banach Function Spaces, Delft, 1955.

[13] LUXEMBURG, W. A. and ZAANEN, A. C. Conjugate Spaces of Orlicz Spaces, Indagat. Math. 59 (1956), 217-228.

[14] LUXEMBURG, W. A. and ZAANEN, A. C. Riesz Spaces I, North-Holland Publ. Comp., Amsterdam-London, 1971.

[15] MATUSZEWSKA, W. and ORLICZ, W. A Note on Modular Spaces. IX., ibidem, 16 (1968), 801-808.

[16] NAKANO, H. Modular Semi-ordered Spaces, Maruzen Co. Ltd., Tokyo, 1950.

[17] NAKANO, H. On Generalized Modular Spaces, Studia Math., 31 (1968), 439-449.

[18] NOWAK, M. The Köthe Dual of Orlicz Spaces Without Local Convexity, Mathematica Japonica (to appear).

[19] ORLICZ, W. On Integral Representability of Linear Functions Over the Space of ϕ-Integrable Functions, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 8 (1960), 567-569.

[20] PERESSINI, A. and SHERBERT, D. R. Order Properties of Liner Mappings of Sequence Spaces, Math. Annalen, 165 (1966), 318-332.

[21] RAO, M. M. Linear Functionals on Orlicz Spaces: General Theory, Pacific J. Math. 25 (1968), 553-585.

[22] ROLEWICZ, S. Metric Linear Spaces, Polish Scientific Publishers, Warszawa, D. Reidel Publ. Comp., 1984.

[23] SCHWARZ, H. U. Banach Lattices and Operators, Tuebner-Texte zur Mathematik 71, Leipzig, 1984.

[24] SHAPIRO, J. H. Extension of Linear Functionals on F-spaces, Duke Math. J. 37 (1970), 639-645.

[25] TURPIN, PH. Convexities dans les Espaces Vectoriels Topologiques Generaux, Dissertationes Math. 131 (1976).

[26] ZAANEN, A. C. Riesz Spaces II, North-Holland Publ. Comp., Amsterdam, New York, 1983.