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EQUICONVERGENCE OF SOME SEQUENCES OF RATIONAL FUNCTIONS

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ABSTRACT. The phenomenon of equiconvergence was first observed by Walsh for two sequences of polynomial interpolants to a class of functions. Here we obtain analogues of Yuanren's results for Walsh equiconvergence using rational functions as in Saff and Sharma. We extend this to Hermite interpolation and an earlier result of [1] is improved and corrected.

KEY WORDS AND PHRASES. Rational functions, polynomials, sequences, interpolation, equiconvergence.

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1. INTRODUCTION.

Walsh equiconvergence theorem is concerned with the class A_ρ of functions $f(z)$ which are analytic in the disk $D_\rho = \{z \mid |z| < \rho\}$ but not analytic in $\overline{D}_\rho$, where $\rho > 1$. If $f(z) \in A_\rho$, the theorem of Walsh asserts that the difference between the Lagrange interpolant on the n roots of unity and the Taylor polynomial of degree $n-1$ about the origin tends to zero as $n \to \infty$ in the disk D_{ρ^2}. Here we shall be interested in the recent extension by Saff and Sharma [2] of Walsh's theorem to rational functions with a given denominator $z^n - \sigma^n$ where σ is a real number > 1. Later the present authors extended these results by replacing Lagrange interpolation with Hermite interpolation. The object of this note is two-fold: We first replace interpolation in the zeros of $z^{m+n+1} - 1$ by interpolation in the zeros of $z^{m+n+1} - \alpha^{m+n+1}$ where $|\alpha| < \rho$ and m is an integer ≥ -1. We also obtain the analogue of Lou Yuanren's recent extension [3] of the Walsh equiconvergence theorem using rational functions in the spirit of Saff and Sharma [2]. Lou [3] was the first to observe that the sum of the ℓ "help polynomials" discussed in [4] have a natural interpretation as the Lagrange interpolant in the nth roots of unity of Taylor polynomial of degree $\ell n - 1$. Secondly we apply the same point of view to Hermite interpolation and thereby improve and correct our earlier results in [1].

2. LAGRANGE INTERPOLATION.

For a fixed integer $m \geq -1$, a positive number $\sigma > 1$ and a number α with $|\alpha| < \rho$, let $R_{n+m,n}(z)$ denote the rational function of the form

$$R_{n+m,n}(z) := \frac{B_{n+m}(z)}{z^n - \sigma^n}, \qquad B_{n+m}(z) \in \Pi_{m+n}(z) \tag{2.1}$$

which interpolates $f(z)$ in the zeros of $z^{m+n+1} - \alpha^{m+n+1}$. Let $r_{n+m,n}(z)$ denote another rational function given by

$$r_{n+m,n}(z) := \frac{P_{n+m}(z)}{z^n - \sigma^n}, \qquad P_{n+m}(z) \in \Pi_{m+n}(z) \tag{2.2}$$

which interpolates $f(z)$ in the zeros of $z^{m+1}(z^n - \sigma^{-n})$. Walsh had shown [5] that if $\sigma > 1$ and $f(z) \in A_\rho$, then

$$\min_{P \in \Pi_{m+n}} \int_{|z|=1} \left| f(z) - \frac{P(z)}{z^n - \sigma^n} \right|^2 |dz| \tag{2.3}$$

is attained when $P(z)$ interpolates $f(z)(z^n - \sigma^n)$ in the zeros of $z^{m+1}(z^n - \sigma^{-n})$. Thus the rational function in (2.2) is characterized by the property (2.3).

It is easy to see that

$$f(z) - R_{n+m,n}(z) = \frac{1}{2\pi i} \int_\Gamma \frac{(z^{m+n+1} - \alpha^{m+n+1})(t^n - \alpha^n)}{(z^n - \sigma^n)(t^{m+n+1} - \alpha^{m+n+1})} \cdot \frac{f(t)}{t - z}\, dt \tag{2.4}$$

where Γ is the circle $|t| = \rho - \varepsilon$ for some small $\varepsilon > 0$. It follows as in [1] that

$$\lim_{n \to \infty} R_{n+m,n}(z) = f(z) \qquad \text{for} \quad |z| < \min(\sigma, \rho) =: \tau.$$

If $K \subset D_\tau$ is compact, then

$$\limsup_{n \to \infty} \|f(z) - R_{n+m,n}(z)\|^{\frac{1}{n}} \leq \frac{1}{\tau} \max\{|\alpha|, \|z\|_K\} \tag{2.5}$$

where $\|\cdot\|_K = \sup_{z \in K} |z|$. Further, if $\rho > \sigma$, then for all $|z| > \sigma$, we have

$$\lim_{n \to \infty} R_{n+m,n}(z) = \begin{cases} 0, & \text{for } m = -1 \\ \sum_{k=0}^{m} a_k z^k, & \text{for } m = 0, 1, 2, \ldots \end{cases} \tag{2.6}$$

where $f(z) = \sum_{k=0}^{\infty} a_k z_k$. From (2.5) and (2.6), we see that Theorem 2.1 in [2] holds also for $R_{n+m,n}(z)$.

Since $f(z) - r_{n+m,n}(z)$ has a representation similar to (2.4), it is easy to see that

$$R_{n+m,n}(z) - r_{n+m,n}(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t - z} [K_1(z,t) + K_2(z,t) + K_3(z,t)]\, dt \tag{2.7}$$

where

$$\begin{cases} K_1(z,t) = \dfrac{t^n - \sigma^n}{z^n - \sigma^n} \cdot \dfrac{\alpha^{m+n+1}}{t^{m+n+1} - \alpha^{m+n+1}}, & K_3(z,t) = \dfrac{t^n - \sigma^n}{z^n - \sigma^n} \cdot \dfrac{(-1) z^{m+1} \sigma^{-n}}{t^n - \sigma^{-n}}, \\ K_2(z,t) = \dfrac{t^n - \sigma^n}{z^n - \sigma^n} \cdot \dfrac{z^{m+n+1}(t^{m+1}\sigma^{-n} - \alpha^{m+n+1})}{t^{m+1}(t^n - \sigma^{-n})(t^{m+n+1} - \alpha^{m+n+1})}. \end{cases} \tag{2.8}$$

From this one can easily prove

THEOREM 1. Let $\rho > 1$ and let $m \geq -1$ be a fixed integer. If $f(z) \in A_\rho$ and if $|\alpha| < \rho$ and $\sigma > 1$, then

$$\lim_{n\to\infty} [R_{n+m,n}(z) - r_{m+n,n}(z)] = 0 \tag{2.9}$$

for

$$|z| < \frac{\rho^2}{\max(|\alpha|, \sigma^{-1})} \qquad \text{when} \quad \sigma \geq \frac{\rho^2}{\max(|\alpha|, \sigma^{-1})}$$

and for $|z| \neq \sigma$ if $\sigma < \rho^2/\max(|\alpha|, \sigma^{-1})$.

For $\alpha = 1$, this theorem gives a result of Saff and Sharma [2]. A slightly more general theorem can be proved if we set $B(z) := z^{\ell(m+1)}(z^{\ell n} - \beta^{\ell n})$, where β is a real number $< \rho$, $|\beta| \neq |\alpha|$ and where $L_{\ell(m+n)}(z, B, f_\sigma)$ denotes the Lagrange interpolant of degree $\ell(m+n+1)-1$ to $f_\sigma(z) := (z^n - \sigma^n)f(z)$ at the zeros of $B(z)$. Let $L_{m+n}(z, \alpha, f_\sigma)$ denote the Lagrange interpolant to f_σ at the zeros of $z^{m+n+1} - \alpha^{m+n+1}$. We shall now prove

THEOREM 2. Let $f(z) \in A_\rho$ $(\rho > 1)$ and let $m \geq -1$ be a fixed integer. If $\alpha \neq \beta$, $(|\alpha|, |\beta| < \rho)$ are given, and if

$$\Delta_{\alpha,B}(z, f) := L_{m+n}(z, \alpha, f_\sigma) - L_{m+n}(z, \alpha, L_{\ell(m+n)}(z, B, f_\sigma)) \tag{2.10}$$

then

$$\lim_{n\to\infty} \frac{\Delta_{\alpha,B}(z, f)}{z^n - \sigma^n} = 0 \tag{2.11}$$

for $|z| < \sigma_1$, if $\sigma > \sigma_1 := \rho^{\ell+1}/\max(|\alpha|, |\beta|)^\ell$ and for $|z| \neq \sigma$ if $\sigma < \sigma_1$. For $\ell = 1$ and $\beta = \sigma^{-1}$, we get Theorem 1.

PROOF. We know that

$$L_{\ell(m+n)}(z, B, f_\sigma) = \frac{1}{2\pi i} \int_\Gamma f_\sigma(t) K(z, t)\, dt$$

where

$$K(z,t) := \frac{B(t) - B(z)}{(t-z)\,B(t)}$$

$$= \frac{1}{B(t)} \left[\frac{t^{\ell(m+n+1)} - z^{\ell(m+n+1)}}{t-z} - \frac{\beta^{\ell n}(t^{\ell(m+1)} - z^{\ell(m+1)})}{t-z} \right].$$

It is easy to see that

$$L_{m+n}(z, \alpha, K(\cdot, t))$$

$$= \frac{1}{B(t)} \left[\frac{t^{\ell(m+n+1)} - \alpha^{\ell(m+n+1)}}{t^{m+n+1} - \alpha^{m+n+1}} \frac{t^{m+n+1} - z^{m+n+1}}{t-z} - \beta^{\ell n} \frac{(t^{\ell(m+1)} - z^{\ell(m+1)})}{t-z} \right].$$

This gives us an integral representation for $\Delta_{\alpha,B}(z, f)$ so that

$$\frac{\Delta_{\alpha,B}(z, f)}{z^n - \sigma^n} = \frac{1}{2\pi i} \int_\Gamma f_\sigma(t) \frac{K_1(z, t)}{z^n - \sigma^n}\, dt \tag{2.12}$$

where

$$K_1(z,t) = \frac{t^{m+n+1} - z^{m+n+1}}{(t-z)(t^{m+n+1} - \alpha^{m+n+1})} \cdot \frac{\alpha^{\ell(m+n+1)} - t^{\ell(m+1)}\beta^{\ell n}}{B(t)}$$

$$+ \frac{\beta^{\ell n}(t^{\ell(m+1)} - z^{\ell(m+1)})}{(t-z)B(t)}. \tag{2.13}$$

From (2.12) and (2.13), we can easily obtain (2.11).

3. HERMITE INTERPOLATION.

Let $f \in A_\rho$ $(\rho > 1)$ and let $\sigma > 1$. For fixed integers m, r, s $(m \geq -1,\ 1 \leq r \leq s)$, set $N = n + m + 1$. Let

$$R_{sN-1}(z,f) := R_{sN-1,nr}(z,f) := C_{sN-1}(z,f)/(z^n - \sigma^n)^r \tag{3.1}$$

where $C_{sN-1}(z,f)$ is a polynomial of degree $\leq sN-1$ which interpolates $f(z)(z^n-\sigma^n)^r$ in the zeros of $(z^N - \alpha^N)^s$, where $|\alpha| < \rho$. For any $\beta \neq \alpha$, $|\beta| < \rho$, let

$$S_{sN-1}(z,f) := S_{sN-1,nr}(z,f) := Q_{sN-1}(z,f)/(z^n - \sigma^n)^r$$

where $Q_{sN-1}(z,f) \in \Pi_{sN-1}$ interpolates $f(z)(z^n-\sigma^n)^r$ in the zeros of $z^{sN-rn}(z^n-\beta^n)^r$. If we set

$$\Delta^{(\alpha,\beta)}_{N,r,s}(z,f) := R_{sN-1}(z,f) - S_{sN-1}(z,f), \tag{3.2}$$

then we shall prove

THEOREM 3. If $f \in A_\rho$ $(\rho > 1)$, and $\alpha \neq \beta$ $(|\alpha|, |\beta| < \rho)$, then $\lim_{n\to\infty} \Delta^{(\alpha,\beta)}_{N,r,s}(z,f) = 0$ in the following situations:

(a) For $|z| < \rho_1 := \rho/\{\frac{\max(|\alpha|,|\beta|)}{\rho}\}^{1/s}$, when $(\frac{\sigma}{\rho})^s > \frac{\rho}{\max(|\alpha|,|\beta|)}$.

(b) For $|z| < \rho_2 := \{\frac{\rho^{s+1}}{\sigma^r \max(|\alpha|,|\beta|)}\}^{1/(s-r)}$, $|z| \neq \sigma$, when $1 < (\frac{\sigma}{\rho})^s < \rho/\max(|\alpha|,|\beta|)$.

(c) For $|z| < \{\frac{\rho^{s-r+1}}{\max(|\alpha|,|\beta|)}\}^{1/(s-r)}$, $|z| \neq \sigma$, when $1 < \sigma < \rho$.

The convergence is uniform and geometric in every compact subset of D.

REMARK. When $\alpha = 1$ and $\beta = 0$, Theorem 3(a) gives Theorem 2.1 in [1], when $s = r$. For $s > r$, cases (b) and (c) in Theorem 3 correct the statement of Theorem 2.1 in [1].

We shall prove a slightly more general result which will yield Theorem 3 when $\ell = 1$.

In the sequel we shall need the following identity ([4]):

$$\left(\frac{z}{t}\right)^{rn} - \left(\frac{z^n-1}{t^n-1}\right)^r = \frac{t^n - z^n}{t^{rn}} \sum_{j=1}^{\infty} \frac{\gamma_{j,r}(z^n)}{t^{jn}}, \tag{3.3}$$

where $\gamma_{j,r}(z^n)$ is a polynomial in z of degree $\leq n(r-1)$, given by

$$\gamma_{j,r}(z^n) = \sum_{k=0}^{r-1} \binom{r+j-1}{k} (z^n-1)^k, \qquad i = 1,2,\ldots. \tag{3.4}$$

In particular, $\gamma_{1,r}(z^n) = z^{rn} - (z^n-1)^r$.

THEOREM 4. If $f(z) \in A_\rho$ $(\rho > 1)$ and if $\alpha \neq \beta$ $(|\alpha|, |\beta| < \rho)$ are any complex numbers, then for any integer $\ell \geq 1$, there exist polynomials $P_{sN-1,j}(z, f)$ of degree $sN-1$ in z $(j = 1, \ldots, \ell - 1)$ depending only on $f(z)(z^n - \sigma^n)^r$ and its power series such that

$$\lim_{n\to\infty} [\Delta^{(\alpha,\beta)}_{N,r,s}(z, f) - \sum_{j=1}^{\ell-1} P_{sN-1,j}(z, f)/(z^n - \sigma^n)^r] = 0 \tag{3.5}$$

for $z \in D$ where $\Delta^{(\alpha,\beta)}_{N,r,s}(z, f)$ is given by (3.2) and the region D is given below:

(a) If $(\frac{\sigma}{\rho})^s \geq (\frac{\rho}{\max(|\alpha|,|\beta|)})^\ell$, then $D = \{z \mid |z| < \rho / \max(\frac{|\alpha|,|\beta|}{\rho})^{\ell/s}$,

(b) If $\rho \leq (\frac{\sigma}{\rho})^s < (\frac{\rho}{\max(|\alpha|,|\beta|)})^\ell$, then

$$D = \{z \mid |z| < \rho\,(\frac{\rho}{\sigma})^{1/s-r} / \max\,(\frac{|\alpha|, |\beta|}{\rho})^{\ell/s-r},\ |z| \neq \sigma\}.$$

(c) If $1 < \sigma < \rho$, then

$$D = \{z \mid |z| < \rho / \max\,(\frac{|\alpha|, |\beta|}{\rho})^{1/s-r}, \quad |z| \neq \sigma\}.$$

The convergence is uniform and geometric in every compact subset of D.

PROOF. It is easy to see that

$$\Delta^{(\alpha,\beta)}_{N,r,s}(z, f) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t - z} \cdot \frac{(t^n - \sigma^n)^r}{(z^n - \sigma^n)^r} \cdot K(t, z)\, dt \tag{3.6}$$

where

$$K(t, z) := \left(\frac{z}{t}\right)^{sN-rn} \left(\frac{z^n - \beta^n}{t^n - \beta^n}\right)^r - \left(\frac{z^N - \alpha^N}{t^N - \alpha^N}\right)^s. \tag{3.7}$$

Rewriting $K(t, z) = K_1(t, z) - K_2(t, z)$, where

$$\begin{cases} K_1(t, z) : = \left(\frac{z}{t}\right)^{sN} - \left(\frac{z^N - \alpha^N}{t^N - \alpha^N}\right)^s, \\ K_2(t, z) : = \left(\frac{z}{t}\right)^{sN} - \left(\frac{z}{t}\right)^{sN-rn} \left(\frac{z^n - \beta^n}{t^n - \beta^n}\right)^r, \end{cases} \tag{3.8}$$

From (3.3), it follows immediately that

$$\begin{cases} K_1(t, z) &= \frac{t^N - z^N}{t^{sN}} \sum_{j=1}^{\infty} \alpha^{(j+s-1)N} \frac{\gamma_{j,s}(z^N \alpha^{-N})}{t^{jN}}, \\ K_2(t, z) &= \left(\frac{z}{t}\right)^{sN-rn} \frac{t^n - z^n}{t^{rn}} \sum_{j=1}^{\infty} \frac{\gamma_{j,r}(z^n \beta^{-n})}{t^{jn}} \gamma^{(j+r-1)n}. \end{cases} \tag{3.9}$$

In [6] it was shown by K. G. Ivanov and Sharma that

$$|\gamma_{j,r}(z^n)| \leq C_0 j^{r-1} \max(1, |z|^{n(r-1)}) \tag{3.10}$$

where C_0 is a constant depending only on r.

From (3.6), (3.8) and (3.9), it follows easily that

$$\Delta^{(\alpha,\beta)}_{N,r,s}(z,f) = \sum_{j=1}^{\infty} P_{sN-1,j}(z,f)/(z^n-\sigma^n)^r \tag{3.11}$$

where $P_{sN-1,j}(z,f)$ are polynomials in z of degree $sN-1$ for each j and are given by

$$P_{sN-1,j}(z,f) := \frac{1}{2\pi i}\int_\Gamma f(t)(t^n-\sigma^n)^r M_j(t,z)\,dt. \tag{3.12}$$

Here $M_j(t,z)$ is a polynomial of degree $sN-1$ in z given by $M_j(t,z) := M_{j,1}(t,z) - M_{j,2}(t,z)$ where

$$M_{j,1}(t,z) := \left(\frac{\alpha}{t}\right)^{(j+s-1)N} \cdot \frac{\gamma_{j,s}(z^N\alpha^{-N})}{t^N} \cdot \frac{t^N - z^N}{t-z} \tag{3.13}$$

and

$$M_{j,2}(t,z) := \left(\frac{\beta}{t}\right)^{(r+j-1)n} \frac{\gamma_{j,r}(z^n\beta^{-n})}{t^n} \cdot \frac{t^n - z^n}{t-z}\left(\frac{z}{t}\right)^{sN-rn}.$$

From (3.4) and (3.13) it is easy to see that $M_j(t,z)$ is a polynomial in z of degree $sN-1$.

For any positive integer $\ell \geq 1$, we see from (3.11) that

$$\Delta^{(\alpha,\beta)}_{N,r,s}(z,f) - \sum_{j=1}^{\ell-1} P_{sN-1,j}(z,f)/(z^n-\sigma^n)^r = \sum_{j=\ell}^{\infty} P_{sN-1,j}(z,f)/(z^n-\sigma^n)^r.$$

In order to estimate the expression on the right above, we see on using (3.10) that for $|z| > \rho$ and $|t| < \rho$, we have

$$\left|\left(\frac{\alpha}{t}\right)^{(j+s-1)N} \frac{\gamma_{j,s}(z^N\alpha^{-N})}{t^N}\,\frac{t^N-z^N}{t-z}\right| \leq |\alpha|^{(j+s-1)N}\,\frac{\max(1,|z|^{N(s-1)}|\alpha|^{-N(s-1)})}{|t|^{sN+jN}}\,|z|^N$$

$$= |\alpha|^{jN}\,\frac{\max(|\alpha|^{(s-1)N},|z|^{(s-1)N})}{|t|^{jN+sN}}\,|z|^N$$

and

$$\left|\left(\frac{\beta}{t}\right)^{(r+j-1)n} \frac{\gamma_{j,r}(z^n\beta^{-n})}{t^n}\cdot\frac{t^n-z^n}{t-z}\cdot\left(\frac{z}{t}\right)^{sN-rn}\right|$$

$$\leq |\beta|^{jn}\,\frac{\max(1,|z|^{n(r-1)}|\beta|^{-n(r-1)})}{|t|^{jn+sN}}\,|\beta|^{(r-1)n}\,|z|^{sN-rn}\,|z|^n$$

$$\leq |\beta|^{jn}\,\frac{\max(|\beta|^{n(r-1)},|z|^{n(r-1)})}{|t|^{jn+sN}}\,|z|^{sN-rn+n}$$

$$\leq \frac{|\beta|^{jn}\,|z|^{sn}}{|t|^{(j+s)n}}\cdot\frac{|z|^{s(m+1)}}{|t|^{s(m+1)}}.$$

Hence with $|t| = R < \rho$ and $|z| > \rho$, we have

$$|(z^n-\sigma^n)|^{-r}\left|\sum_{j=\ell}^{\infty} P_{sN-1,j}(z,f)\right| \leq C\left|\frac{R^n-\sigma^n}{|z|^n-\sigma^n}\right|^r \sum_{j=\ell}^{\infty}\left(\frac{|\alpha|^{jN}|z|^{sN}}{|R|^{jN+sN}}\,j^{s-1} + \frac{|\beta|^{jn}|z|^{sN}}{R^{jn+sN}}\,j^{r-1}\right)$$

$$\leq C\left|\frac{R^n-\sigma^n}{|z|^n-\sigma^n}\right|^r \frac{|z|^{sn}|z|^{s(m+1)}\max(|\alpha|^{\ell n},|\beta|^{\ell n})}{R^{(s+\ell)n}R^{(s+\ell)(m+1)}}$$

$$\times\sum_{j=0}^{\infty}\left\{(j+\ell)^{s-1}\left(\frac{|\alpha|}{R}\right)^{jn}+(j+\ell)^{r-1}\left(\frac{|\beta|}{R}\right)^{jn}\right\}$$

$$\leq C_1\left|\frac{R^n-\sigma^n}{|z|^n-\sigma^n}\right|^r \frac{|z|^{sn}\max(|\alpha|^{\ell n},|\beta|^{\ell n})}{R^{(s+\ell)n}}.$$

From the above we see that

$$\lim_{n\to\infty}\left[\Delta^{(\alpha,\beta)}_{N,r,s}(z,f)-\sum_{j=1}^{\ell-1}\frac{P_{sN-1,j}(z,f)}{(z^n-\sigma^n)^r}\right]=0 \tag{3.14}$$

when

$$|z|<\rho_1 \qquad \text{if} \quad \sigma>\rho_1:=\left\{\frac{R^{s+\ell}}{\max(|\alpha|^{\ell},|\beta|^{\ell})}\right\}^{1/s}. \tag{3.15}$$

If $\rho<\sigma<\rho_1$, then (3.14) holds if

$$|z|<\rho_2:=\left\{\frac{R^{s+\ell}}{\sigma^r\max(|\alpha|^{\ell},|\beta|^{\ell})}\right\}^{1/(s-r)} \qquad \text{and} \qquad |z|\neq\sigma. \tag{3.16}$$

If $\sigma<\rho$, then (3.14) holds if

$$|z|<\rho_3:=\left\{\frac{R^{s+\ell-r}}{\max(|\alpha|^{\ell},|\beta|^{\ell})}\right\}^{1/(s-r)} \qquad \text{and} \qquad |z|\neq\sigma. \tag{3.17}$$

It is easy to see that (3.15), (3.16) and (3.17) give the region D in the cases (a), (b), (c) of Theorem 4.

This completes the proof of the Theorem.

It may be noted that $\rho_1>\rho$ and $\rho_3>\rho$, but $\rho_2>\rho$ only if

$$\frac{\rho^{\ell}}{\max(|\alpha|^{\ell},|\beta|^{\ell})}>\left(\frac{\sigma}{\rho}\right)^r$$

and $\rho_2>\sigma$ if $(\frac{\sigma}{\rho})^s<\frac{\rho^{\ell}}{\max(|\alpha|^{\ell},|\beta|^{\ell})}$. When $s=r$, both (3.16) and (3.17) give the result that convergence holds for $|z|\neq\sigma$. When $\ell=1$, we get the result of Theorem 3.

REMARK. The "help functions" $P_{sN-1,j}(z,f)$ in Theorem 4 seem to be different from those obtained in [1] in Theorem 2.2. When $s\neq r$, Theorem 4 provides the correction to Theorem 2.2 in our earlier paper [1].

It would be interesting to interpret the polynomials $P_{sN-1,j}(z,f)$ $(j=1,\ldots,\ell-1)$ as having some interpolaory property as has been done in [6] and [7]. If we set

$$f_\sigma(z):=f(z)(z^n-\sigma^n)^r$$

and if $T_{n-1}f_\sigma$ denotes the Taylor polynomial of f_σ of degree $\leq n-1$, we shall show that

$$\sum_{j=1}^{\ell-1}P_{sN-1,j}(z,f)=H_{sN-1}\big(z,\alpha,(T_{(s+\ell-1)N-1}-T_{sN-1})f_\sigma\big)$$

$$-z^{sN-rn}H_{rn-1}(z,\beta,z^{rn}T_{\ell n-n-1}f_\sigma) \tag{3.18}$$

where $H_{sN-1}(z,\alpha,g)$ denotes the Hermite interpolant of degree $sN-1$ to g on the zeros of $(z^N-\alpha^N)^s$. Recalling that if $g_\ell(z) := z^{j+(s+\ell-1)N}$, where $0 \le j \le n-1$, then

$$H_{sN-1}(z,\alpha,g_\ell) = z^j \alpha^{N(s+\ell-1)} \gamma_{\ell,s}(z^N\alpha^{-N}),$$

we can verify that

$$H_{sN-1}\Big(z,\alpha,\ \frac{z^{sN}(t^{\ell N-N}-z^{\ell N-N})}{t^{(s+\ell-1)N}(t-z)}\Big) = \frac{t^N-z^N}{t-z}\cdot\frac{1}{t^{sN}}\sum_{j=1}^{\ell-1}\frac{\gamma_{j,s}(z^N\alpha^{-N})}{t^{jN}}\,\alpha^{N(s+j-1)}.$$

Similarly,

$$H_{rn-1}\big(z,\beta,\ \frac{z^{rn}(t^{\ell n-n}-z^{\ell n-n})}{t^{sn+\ell n-n}(t-z)} = \frac{t^n-z^n}{t-z}\cdot\frac{1}{t^{rn}}\sum_{j=1}^{\ell-1}\frac{\gamma_{j,r}(z^n\beta^{-n})}{t^{jn}}\,\beta^{n(r+j-1)}.$$

From the above we can get (3.18) easily. Formula (3.18) is not exactly of the same kind as the corresponding results in [7] and [3]. However it provides an interpretation for the "help functions" in terms of iteration of interpolation operators. It would be interesting to see if the regions of equiconvergence can be extended by application of summability methods as has been done in [8] by R. Bruck for the case of Lagrange interpolation.

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