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FIXED POINT THEOREMS FOR d-COMPLETE TOPOLOGICAL SPACES I

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ABSTRACT. Generalizations of Banach's fixed point theorem are proved for a large class of non-metric spaces. These include d-complete symmetric (semi-metric) spaces and complete quasi-metric spaces. The distance function used need not be symmetric and need not satisfy the triangular inequality.

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Let (X,t) be a topological space and $d : X \times X \to [0,\infty)$ such that $d(x,y) = 0$ if and only if $x = y$. X is said to be d-complete if $\sum_{n=1}^{\infty} d(x_n,x_{n+1}) < \infty$ implies that the sequence $\{x_n\}$ is convergent in (X,t). In a metric space, such a sequence is a Cauchy sequence. If $T : X \to X$, $O(x,\infty) = \{x,Tx,T^2x,\cdots\}$ is called the orbit of x. $G : X \to [0,\infty)$ is T-orbitally lower semi-continuous at x^* if $\{x_n\}$ is a sequence in $O(x,\infty)$ and $\lim x_n = x^*$ implies $G(x^*) \le \lim\inf G(x_n)$. $T: X \to X$ is w-continuous at x if $x_n \to x$ implies $Tx_n \to Tx$.

The basic idea of a d-complete topological space goes back to Kasahara [1] and [2], Iseki [3], and their L-spaces.

LEMMA 1. X is a set, $T : X \to X$ and $d : X \times X \to [0,\infty)$. Then there exists $\phi : X \to [0,\infty)$ such that

(a) $d(x,Tx) \le \phi(x) - \phi(Tx)$ for all $x \in X$, if and only if

(b) $\sum_{n=0}^{\infty} d(T^nx,T^{n+1}x)$ converges for all x.

(c) If (a) holds for all $y \in O(x,\infty)$, then $\sum_{n=0}^{\infty} d(T^nx,T^{n+1}x)$ converges.

PROOF. Suppose (a) holds. For $x \in X$, $S_n = \sum_{i=0}^{n} d(T^ix,T^{i+1}x) \le \sum_{i=0}^{n} [\phi(T^ix) - \phi(T^{i+1}x)] = \phi(x) - \phi(T^{n+1}x) \le \phi(x)$. $\{S_n\}$ is non-decreasing and bounded above and therefore convergent.

Suppose (b) holds and let $\phi(x) = \sum_{i=0}^{\infty} d(T^i x, T^{i+1}x)$. $\phi(x) - \phi(Tx) = d(x,Tx)$ since $S_n - T_n = \sum_{i=0}^{n} d(T^i x, T^{i+1}x) - \sum_{i=0}^{n} d(T^{i+1}x, T^{i+2}x) = d(x,Tx) - d(T^{n+1}x, T^{n+2}x) \to d(x,Tx)$ as $n \to \infty$.

The proof of (b) gives (c).

LEMMA 2. X is a topological space, $T : X \to X$ and $d : X \times X \to [0,\infty)$ such that $d(x,y) = 0$ if and only if $x = y$. Suppose there exists an $x \in X$ such that $\lim T^n x = x^*$ exists. Then:

(a) $Tx^* = x^*$ implies $G(x) = d(x,Tx)$ is T-orbitally lower semi-continuous at x^*.

(b) G is T-orbitally lower semi-continuous at x^* and $\lim\inf d(T^n x, T^{n+1}x) = 0$ imply $Tx^* = x^*$.

PROOF. Assume that $Tx^* = x^*$ and (x_n) is a sequence in $O(x,\infty)$ with $\lim x_n = x^*$. Then $G(x^*) = d(x^*,Tx^*) = 0 \le \lim\inf d(x_n,Tx_n) = \lim\inf G(x_n)$.

If $x_n = T^n x \to x^*$ and G is T-orbitally lower semi-continuous at x^*, then $0 \le d(x^*,Tx^*) = G(x^*) \le \lim\inf G(x_n) = \lim\inf d(T^n x, T^{n+1}x) = 0$. Thus, $Tx^* = x^*$.

The next theorem is a version of Caristi's theorem [4 or 5] in this more general setting. Caristi's theorem for metric spaces is a generalization of Banach's fixed point theorem.

THEOREM 1. Let X be a d-complete topological space. Suppose $T : X \to X$ and $\phi : X \to [0,\infty)$. Suppose there exists an x such that

$$d(y,Ty) \le \phi(y) - \phi(Ty) \text{ for all } y \in O(x,\infty). \tag{1}$$

Then we have:

(a) $\lim T^n x = x^*$ exists.

(b) $Tx^* = x^*$ if and only if $G(x) = d(x,Tx)$ is T-orbitally lower semi-continuous at x.

PROOF. From Lemma 1, $\sum_{n=0}^{\infty} d(T^n x, T^{n+1}x)$ is convergent. X is d-complete so $\lim T^n x = x^*$ exists. Note that $\lim d(T^n x, T^{n+1}x) = 0$. Now apply Lemma 2.

Even in this general setting one obtains a version of Banach's theorem as a corollary of the above theorem.

COROLLARY 1. Let X be a d-complete topological space and $0 < k < 1$. Suppose $T : X \to X$ and there exists an x such that

$$d(Ty,T^2y) \le k\, d(y,Ty) \text{ for all } y \in O(x,\infty). \tag{2}$$

Then:

(a) $\lim T^n x = x^*$ exists.

(b) $Tx^* = x^*$ if and only if $G(x) = d(x,Tx)$ is T-orbitally lower semi-continuous at x.

PROOF. Set $\phi(y) = \frac{1}{1-k} d(y,Ty)$ for $y \epsilon O(x,\infty)$. Let $y = T^n x$ in (2). Then $d(T^{n+1}x,T^{n+2}x) \leq k\, d(T^n x,T^{n+1}x)$ and $d(T^n x,T^{n+1}x) - k\, d(T^n x,T^{n+1}x) \leq d(T^n x,T^{n+1}x) - d(T^{n+1}x,T^{n+2}x)$. Thus, $d(T^n x,T^{n+1}x) \leq \frac{1}{1-k} [d(T^n x,T^{n+1}x) - d(T^{n+1}x,T^{n+2}x)]$ or $d(y,Ty) \leq \phi(y) - \phi(Ty)$. Apply Theorem 1.

In [6], it was shown that many generalizations of Banach's theorem hold for quasi-metric ($d(x,y) \neq d(y,x)$) spaces. However, we do not have the triangular inequality in the present setting, so standard proofs and theorems do not necessarily hold. Before proving more theorems, we give the definitions of some of the special d-complete topological spaces we had in mind when we formulated the results of this paper.

DEFINITION. A symmetric on a set X is a real-valued function d on $X \times X$ such that:

(a) $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$; and

(b) $d(x,y) = d(y,x)$.

Let d be a symmetric on a set X and for any $\epsilon > 0$ and any $x \epsilon X$, let $S(x,\epsilon) = \{y \epsilon X: d(x,y) < \epsilon\}$. We define a topology t(d) on X by $U \epsilon t(d)$ if and only if for each $x \epsilon U$, some $S(x,\epsilon) \subset U$. A symmetric d is a semi-metric if for each $x \epsilon X$ and each $\epsilon > 0$, $S(x,\epsilon)$ is a neighborhood of x in the topology t(d). A topological space X is said to be symmetrizable (semi-metrizable) if its topology is induced by a symmetric (semi-metric) on X.

THEOREM 2. Suppose X is a d-complete Hausdorff topological space, $T : X \to X$ is w-continuous and satisfies $d(Tx,T^2x) \leq k(d(x,Tx))$ for all $x \epsilon X$, where $k:[0,\infty) \to [0,\infty)$, $k(0) = 0$, and k is non-decreasing. Then T has a fixed point if and only if there exists an x in X with $\sum_{n=1}^{\infty} k^n(d(x,Tx)) < \infty$. In this case, $x_n = T^n x \to p = TP$. [k is not assumed to be continuous and $k^2(a) = k(k(a))$].

PROOF. If $Tp = p$, $d(p,Tp) = 0$, and $k^n(0) = 0$ for every n.

Assume the condition holds and let $x_n = T^n x$. $d(Tx,T^2x) \leq k(d(x,Tx))$ and $d(T^2x,T^3x) \leq k(d(Tx,T^2x)) \leq k^2(d(x,Tx))$. By induction,

$$d(x_n,x_{n+1}) = d(T^n x,T^{n+1}x) \leq k^n(d(x,Tx)).$$

Now $\sum_{n=1}^{\infty} d(x_n,x_{n+1}) < \infty$ and X is d-complete. Thus, $\lim x_n = p$ exists. T w-continuous implies $x_{n+1} = T\, x_n \to Tp$. Since $x_{n+1} \to p$, $Tp = p$.

In many special cases of Theorem 2, one has $d(Tx,Ty) \leq k(d(x,y))$ for all $x, y \epsilon X$ and the special form of k forces w-continuity of T. It may also force the uniqueness of p and enable us to obtain error bounds. In other cases, T need not be w-continuous so the following theorem is needed.

THEOREM 3. Suppose X is a d-complete Hausdorff topological space, $T : X \to X$ and there exists a y such that $d(Tx,T^2x) \leq k(d(x,Tx))$ for every $x \epsilon O(y,\infty)$, where $k : [0,\infty) \to [0,\infty)$, $k(0) = 0$, and k is non-decreasing. Suppose $\sum_{n=1}^{\infty} k^n(d(y,Ty)) < \infty$

and (x_n) in $O(y,\infty)$ with $\sum_{n=1}^{\infty} d(x_n,x_{n+1}) < \infty$ imply $\lim x_n$ exists. Let $x_n = T^n y$. Then:

(a) $\lim x_n = p$ exists.

(b) $Tp = p$ if and only if $G(x) = d(x,Tx)$ is T-orbitally lower semi-continuous at p.

PROOF. The proof of Theorem 2 gives (a). Lemma 2 gives (b) since $0 = \lim_n d(x_n,x_{n+1}) = \lim_n d(T^n y,T^{n+1}y) = 0$.

COROLLARY 2. Let $0 < \lambda < 1$. Suppose X is a d-complete Hausdorff topological space, $T : X \to X$ is w-continuous and satisfies $d(Tx,Ty) \le \lambda\, d(x,y)$ for all $x, y \in X$. Then T has a unique fixed point p. For any $x \in X$, $p = \lim T^n x$.

PROOF. Let $k(t) = \lambda t$. Then $k^n(d(x,Tx)) = \lambda^n d(x,Tx)$, so $\sum_{n=1}^{\infty} k^n(d(x,Tx)) < \infty$ for any x in X. T is w-continuous so Theorem 2 gives $x_n = T^n x \to p = Tp$ for any $x \in X$. Clearly, p is unique since $0 < \lambda < 1$.

In Corollary 2, if one replaces topological by symmetrizable, then $d(Tx,Ty) \le \lambda\, d(x,y)$ forces T to be w-continuous. We now give several examples of a specific function k where Theorem 2 or Theorem 3 applies to yield another corollary similar to Corollary 2. To apply the theorems, one needs a non-decreasing function k and an x in X with $\sum_{n=1}^{\infty} k^n(d(x,Tx)) < \infty$. The following examples satisfy these conditions. The reader can consult [7] for the details that are not obvious and not provided here.

EXAMPLE 1. Suppose $0 < \lambda < 1$. Let $k(t) = \lambda t$ for $t \ge 0$. Then $d(Tx,T^2x) \le k(d(x,Tx)) = k\, d(x,Tx)$ gives $k^n(d(x,Tx)) = \lambda^n d(x,Tx)$.

EXAMPLE 2. Suppose T satisfies $d(Tx,Ty) \le \phi(d(x,y))\, d(x,y)$ for all x,y in X, where ϕ: $[0,\infty) \to [0,1)$ and ϕ is non-decreasing. Then $k(t) = t\,\phi(t)$, k is non-decreasing, and $k : [0,\infty) \to [0,\infty)$. It follows by induction that $k^n(t) \le t[\phi(t)]^n$. Since $\phi(t) < 1$, $\sum_{n=1}^{\infty} k^n(t) < \infty$.

EXAMPLE 3. Consider $k(t) = t\,\phi(t)$ where $\phi : [0,\infty) \to [0,\infty)$ and $\phi(t) \le t$ for $t < 1$. If $t < 1$, it follows that $k^n(t) \le t[\phi(t)]^n$. If k is non-decreasing, the theorems apply.

EXAMPLE 4. $k(t) = t\,\phi(t)$ where $\phi : [0,\infty) \to [0,\infty)$ and $\phi(\alpha t) \le \alpha\,\phi(t)$ for $\alpha \in (0,1]$. If $\phi(t) < 1$, $k^n(t) \le (kt)(\phi(t))^n$ for all $n \ge 2$. If k is non-decreasing, the theorems apply.

EXAMPLE 5. Assume k is non-decreasing, k is convex on [0,1], and $k(t) < t$ for all $0 < t < 1$. Fix $t < 1$. Now $k(t) < t$ gives $k(t) = \alpha t$ for some $0 < \alpha = \alpha(t) < 1$. By induction, $k^n(t) \le \alpha^n t$ for all n and thus $\sum_{n=1}^{\infty} k^n(t) < \infty$.

THEOREM 4. Suppose (X,d) is a Hausdorff d-complete symmetrizable space,

$T : X \to X$, and $d(Tx,Ty) \le [d(x,y)]^p$ where $p > 1$. If there exists x such that $\mu = d(x,Tx) < 1$, then $x_n = T^n x \to p = Tp$.

PROOF. Let $k(t) = t^p$ for $t \ge 0$. $k(0) = 0$, k is increasing, $k(t) < t$ if $t < 1$, and k is convex. Since $0 \le \mu < 1$, $\sum_{n=1}^{\infty} k^n(\mu) < \infty$. T is w-continuous so Theorem 2 applies.

REMARKS. Given a topological space (X,t), when does there exist a distance function d such that X is d-complete? Let X be an infinite set and t any T_1 non-discrete first countable topology for X. Then there exists a complete metric d for X such that $t \le t_d$ and the metric topology t_d is non-discrete. Now (X,t) is d-complete since $\Sigma\, d(x_n,x_{n+1}) < \infty$ implies that $\{x_n\}$ is a d-Cauchy sequence. Thus, $x_n \to x$ in t_d and therefore in the topology t. In [8], the construction gives $t \le t_E$ where t_E is a complete uniform space and the uniformity has a countable base. Hence, the uniformity is metrizable and the compatible metric d must be complete.

It should also be noted that any complete quasi-metric space (X,d) ($d(x,y) \ne d(y,x)$) is a d-complete topological space. There are several competing definitions for a Cauchy sequence, but $\Sigma d(x_n,x_{n+1}) < \infty$ will imply that $\{x_n\}$ is a Cauchy sequence for any reasonable definition. One reasonable definition is obtained by requiring that the filter generated by $\{x_n\}$ be a Cauchy filter in the quasi-uniformity generated by d. This gives $\{x_n\}$ is a Cauchy sequence if for each $\epsilon > 0$ there exists a positive integer $n_0 = n(\epsilon)$ and $x = x(\epsilon)$ in X such that $\{x_n: n \ge n_0\} \subseteq \{y \in X: d(x,y) < \epsilon\}$. The metric space definition of a Cauchy sequence also holds if $\Sigma d(x_n,x_{n+1}) < \infty$.

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