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FOURIER TRANSFORMS OF DINI-LIPSCHITZ FUNCTIONS ON VILENKIN GROUPS

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ABSTRACT. In [4] we proved some theorems on the Fourier Transforms of functions satisfying conditions related to the Dini-Lipschitz conditions on the n-dimensional Euclidean space R^n and the torus group T^n. In this paper we extend those theorems for functions with Fourier series on Vilenkin groups.

KEY WORDS AND PHRASES. Dini-Lipschitz Functions, Vilenkin Fourier Series.
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1. INTRODUCTION.

Let $f(x)$ belong to the Lebesgue space $L^p, 1 < p \leq 2$ of functions on the real line R or on the circle group T with its usual norm $\| \cdot \|_p$. The pth integral modulus of continuity $\omega_p(f,h)$ is defined as

$$\omega_p(f,h) = \sup \| f(x+h) - f(x) \|_p .$$

In [4] (Theorem 3.3) we proved that if $f(x)$ belongs to $L^p(R)$ such that

$$\omega_p(f,h) = 0(h^\alpha/(\mathrm{Log}\ h)^\gamma, \quad o < \alpha \leq 1$$

then the Fourier Transform $\widehat{f}$ belongs to $L^\beta(R)$ where

$$p/(p+\alpha p-1) < \beta \leq p' = p/(p-1), \quad \beta > 1/\gamma' .$$

In the present work we shall extend this result for functions on $L^p(G)$ where G is a compact metrizable zero-dimensional Abelian group, i.e., a Vilenkin group.

2. DEFINITIONS AND NOTATIONS.

Here we introduce some definitions and notations that will be used later on. This is briefly done since we shall mainly follow Onneweer [1] and Quek and Yap [2] in this respect.

Let G be a Vilenkin group. Then its dual $\widehat{G}$ is a discrete countable torsion group. It is well known that one can introduce on G a countable basic set of neighbourhoods $\{G_n\}$ of the identity element $\{e\}$ of G such that

$$G = G_o \supset G_1, \supset G_2, ..., \text{ and } \bigcap_{n=o}^{\infty} G_n = \{e\} .$$

On the other hand, let V_n denote the annihilator in $\widehat{G}$ of the subgroup G_n in G. Then it is known that

$$\{e\} = V_o \subset V_1, ..., \subset \widehat{G}, \text{ and that } \bigcup_{n=o}^{\infty} V_n = \widehat{G}\ .$$

If all V_n are finite, the inclusion is proper. We introduce the numbers $m_o, m_1, m_2, ..., m_k$ such that

$$m_o = 1, m_{k+1} = p_k m_k; \quad k \varepsilon N,$$

p_k being a prime ≥ 2. Then every V_n has m_n as its measure and the quotient subgroup V_n / V_{n-1} has P_n for its measure.

DEFINITION 2.1. For $x \varepsilon G$, let (n, x) denote the continuous character of x, i.e. $(n, x) \varepsilon \widehat{G}$. The Fourier transform $\widehat{f}(n)$ of $f(x) \varepsilon L^p(G)$ is defined by

$$\widehat{f}(n) = \int_G f(x)\overline{(n,x)}dx$$

where $\overline{(n,x)}$ is the complex conjugate of (n, x).

DEFINITION 2.2. Let $f(x) \varepsilon L^p(G)$. The pth modulus of continuity $\omega_p(f, k)$ is defined by

$$\omega_p(f,k) = \sup_{h \varepsilon G_k} \| f(x+h) - f(x) \|_p$$

The Lipschitz and the Dini-Lipschitz classes in $L^p(G)$ are those for which $\omega_p(f, k) = 0\ (m_k^{-\alpha})$ and $o \mid \text{Log } m_k \mid^{-1}$ respectively.

DEFINITION 2.3 If every P_k is finite as $k \to \infty$ we say that G has the boundedness property (P).

3. MAIN RESULTS.

With the previous definitions and notations in hand, we now prove the analogue of Theorem 3.3 in [4]. Thus we state the following

THEOREM 3.1. Le $f(x) \varepsilon L^p(G)$, $1 < p \leq 2$, such that

$$\omega_p(f,k) = 0(m_k^{-\alpha}/(\text{Log } m_k)^{\gamma}), \quad o < \alpha \leq 1\ . \tag{3.1}$$

Then $\widehat{f}(n) \varepsilon 1^{\beta}(\widehat{G})$ for $q = P/(p-1) \geq \beta > \max(p/(p+\alpha p - 1),\ 1/\gamma)$.

PROOF. Since the Fourier transform of $f(x+h) - f(x)$ is given by $\widehat{f}(n)\ (n,h) - 1)$, the Hausdorff-Young theorem yields

$$\sum_{\widehat{G}} \mid \widehat{f}(n) \mid^q \mid (n,h) - 1 \mid^q \qquad < M\omega_p(f,k)^q = 0(m_k^{-\alpha q}/(\text{Log} m_k)^{\gamma q})\ .$$

The boundedness property (P) for G gives (see Onneweer [1], (2)).

$$\sum_{n=m_k}^{m_{k+1}-1} \mid \widehat{f}(n) \mid^q = 0(m_k^{-\alpha q}/\ (\text{Log } m_k)^{\gamma q}$$

Applying the Holder's inequality with $\beta \leq q$ for the last estimate one arrives at

$$\sum_{n=m_k}^{m_{k+1}-1} \mid f(n) \mid^{\beta} = 0(m_k^{-\alpha\beta}/(\text{Log } m_k)^{\gamma\beta})\ (m_k^{1-\beta/q})$$

and this leads to the final estimate

$$\sum_{n \varepsilon \widehat{G}}^{\infty} \mid \widehat{f}(n) \mid^{\beta} = 0\ (\sum_{k=0}^{\infty} (m_k^{1-a\beta-\beta+\beta/p})\ (\text{Log } m_k^{-\gamma\beta})\ .$$

If $1-\alpha\beta-\beta+\beta/_p<o$ and $-\gamma\beta<-1$ the series is convergent since $m_k\geq 2^k$, and this proves the theorem.

REMARK 3.2. We remark here that for special choice of α,γ, and P like. $\alpha=1$, $\gamma=1$, $P=2$, the previous theorem gives special interesting cases. This is quite obvious and we shall not deal with it any further. However, the special case $P=2$ and $o<\alpha<1$ is particularly important and deserves special consideration.

4. FUNCTIONS IN $L^2(G)$. The origin of this section is a theorem proved in Titchmarsh ([3] Theorem 85, p. 117) for functions belonging to Lip$(\alpha,2)$ on the real line R. For further reference we state it as.

THEOREM 4.1. Let $f(x)\varepsilon$ Lip $(\alpha,2)$ on R. Then the conditions

$$\omega_2(f,h)=0(h^\alpha)\,,\quad 0<\alpha<1,\quad h\to o$$

and

$$\left[\int_{-\infty}^{-X}+\int_X^\infty\right]|\hat{f}|^2\,\mathrm{du}\;=0(X^{-2\alpha}),\text{ as } X\to\infty$$

are equivalent.

This theorem was studied rather extensively in [5] and [6] for functions in $L^2(R^2)$, and $L^2(T^2)$ respectively, where several conditions of the order of magnitude for the Fourier transforms $\hat{f}$ of f proved to be equivalent to one another.

In [4] (Theorems 5.1, 5.2) we proved an analogue of Theorem 4.1 for the Dini-Lipschitz functions in $L^2(R)$. In this section we shall prove Theorem 5.2 in [4] for functions in $L^2(G)$.

THEOREM 4.2. Let $f(x)$ belong to $L^2(G)$. Then the conditions

$$\omega_2(f,k)=0(h^\alpha/(\mathrm{Log}\;h)^\gamma),\quad h\varepsilon G_k \tag{4.1}$$

and

$$\sum_{n=m_k}^{\infty}|\hat{f}(n)|^2=0(m_k^{-2\alpha}/(\mathrm{Log}(m_k^2)^\gamma) \tag{4.2}$$

are equivalent. Here $h=m_k^{-1}$.

PROOF. That the first implication is true follows from Theorem 3.1 where it is proved that

$$\sum_{n=m_k}^{\infty}|\hat{f}(n)|^q=0(\omega_p(f,k))^q$$

Taking $p=q=2$ and substituting for $h=m_k^{-1}$ we obtain (4.2). We also hint that an argument based on the Parseval's identity similar to that of Titchmarsh's leads independently to the same result. To prove the converse let (4.2) hold. Then

$$\sum_{n=m_k}^{m_{k+1}-1}|\hat{f}(n)|^2=0(m_k^{-2\alpha}/(\mathrm{Log}m_k^{2\gamma}-0(m_{k+1}^{-2\alpha}/(\mathrm{Log}\;m_{k+1})^{2\gamma}) \tag{4.3}$$

Since G has the boundedness property (P); hence every $P_k=m_{k+1}/m_k$ is finite for all $k\varepsilon N$, the same is true of LogP_k. Thus the right hand sides of (4.2) and (4.3) are the same. This applies to estimates of the form

$$\sum_{n=m_k}^{m_{k+1}-1}|\hat{f}(n)|^2\quad|(n,h)-1|^2$$

To sum up, by the Parseval's identity one obtains

$$\int_G |f(x+h)-f(x)|^2 dx = \sum_{\widehat{G}} |\hat{f}(n)|^2 |(n,h)-1|^2$$

$$= \sum_{m_k}^{\infty} + \sum_{o}^{m_k} = 0\,|m_k^{-2\alpha}/\,(\mathrm{Log}\, m_k)^{2\gamma}|$$

This is equivalent to (4.1) upon substituting for $h = m_k^{-1}$, $h \epsilon G_k$ and the proof is complete.

REMARK 4.2. We conclude by indicating that Theorem 5.1 in [4] is true for Vilenkin Fourier series, since, it can be deduced as a special case of Theorem 4.1. We also add that for $0 < \alpha < 1$, Theorems 3.1 and 4.1 of the present paper can be proved for higher differences of $f(x) \epsilon L^p(G)$. The statements and the proofs are almost straightforward and will not be given.

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