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INTEGRAL REPRESENTATIONS OF GENERALIZED LAURICELLA HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. The generalized hypergeometric function was introduced by Srivastava and Daoust. In the present paper a new integral representation is derived. Similarly new integral representations of Lauricella and Appell function are obtained.

KEY WORDS. Lauricella, Appell, integral representations, Mellin transformation.

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1. INTRODUCTION.

The generalized Lauricella function of several complex variables was introduced by Srivastava and Daoust [7,8]. The only integral representation which seems to be known for this function is in terms of a Mellin-Barnes integral [6,8]. From a fundamental result about Mellin transforms in n-dimensions, we obtain a new representation for the generalized Lauricella function under suitable restrictions on the parameters. In a similar manner new integral representations are obtained for the Lauricella functions $F_A^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$, and consequently for the Appell functions F_1, F_2, and F_4. From these derivations it is clear that the method does not provide representations for $F_B^{(n)}$ and F_3.

2. THE FUNDAMENTAL THEOREM.

Although the following theorem is quite simple, nevertheless it has basic importance.

THEOREM. *Let $f(x)x^{s-1} \in L(0,\infty)$ and*

$$M_1[f(x)](s) = \int_0^\infty f(x)x^{s-1}dx = f^*(s) \ . \tag{1}$$

If $Re s_j > 0$, $j = 1,2,\cdots,n$ and $Re\left(\sum_{j=1}^n s_j\right) = Re s$, then

$$M_n[f(\max\{x_1,\cdots,x_n\})](s_1,\cdots,s_n) = \int_0^\infty \cdots \int_0^\infty f(\max\{x_1,\cdots,x_n\}) \prod_{i=1}^n x_i^{s_i-1}dx_1\cdots dx_n = \tag{2}$$
$$= \frac{s_1+\cdots+s_n}{s_1\cdots s_n} f^*(s_1+\cdots+s_n) \ .$$

Here M_n represents the n-dimensional Mellin integral transformation [2].

PROOF. In fact we have

$$\int_0^\infty \cdots \int_0^\infty f(\max\{x_1,\cdot s,x_n\}) \prod_{i=1}^n x_i^{s_i-1}dx_1\cdots dx_n =$$
$$= \sum_{i=1}^n \int_0^\infty f(x_i) \int_0^{x_i} \overset{(n-1)}{\cdots} \int_0^{x_i} \prod_{j=1}^n x_j^{s_j-1}dx_1\cdots dx_n =$$

$$= \sum_{i=1}^{n} \frac{s_i}{s_1 s_2 \cdots s_n} \int_0^\infty f(x_i) x_i^{s_1 + \cdots + s_n - 1} dx_i = \frac{s_1 + \cdots + s_n}{s_1 \cdots s_n} f^*(s_1 + \cdots + s_n) .$$

Thus the formula (2) is proved.

The only related results which we have found in the literature are those for the two-dimensional Laplace transformation in the tables of Voelker and Doetsch [9; p. 165, (30),(32)] and in the work of Černov [4; p. 145]. In an analogous manner to our theorem, those results easily can be derived and extended to higher dimensions.

3. THE INTEGRAL REPRESENTATION OF THE GENERALIZED LAURICELLA FUNCTION.

The generalized hypergeometric function of Srivastava and Daoust [6] is defined by

$$F^{p\, p_1, \cdots, p_n}_{q\cdot q_1, \cdots, q_n}\left({a_p ; c^1_{p_1}; \cdots ; c^n_{p_n} ; \atop b_q ; d^1_{q_1}; \cdots ; d^n_{q_n} ;} x_1, \cdots, x_n \right) = \tag{3}$$

$$= \sum_{m_1, \cdots, m_n = 0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{m_1 + \cdots + m_n}}{\prod_{j=1}^{q} (b_j)_{m_1 + \cdots + m_n}} \prod_{i=1}^{n} \left\{ \frac{\prod_{j=1}^{p_i} (c^i_j)_{m_i} x_i^{m_i}}{\prod_{j=1}^{q_i} (d^i_j)_{m_i} m_i!} \right\}$$

where for absolute convergence it is sufficient that

$$1 + q + q_k - p - p_k \geq 0 ; \qquad k = 1, 2, \cdots, n .$$

The function in (3) is a special case of the H-functions of several variables which were defined in [3,6]. The fundamental theorem leads to the following result which involves the G-function of Meijer [8]. Let

$$f(x) = G^{p\,0}_{q\,p}\left(x \mid {b_1, \cdots, b_q \atop a_1, \cdots, a_p} \right) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + s)}{\prod_{j=1}^{q} \Gamma(b_j + s)} x^{-s} ds$$

where $\mathrm{Re} a_j > 0$, $j = 1, 2, \cdots, p$; and either $p = q$ and $\mathrm{Re} \sum_{j=1}^{p} a_j < \mathrm{Re} \sum_{j=1}^{q} b_j$ or $q < p$. It is known [5] that

$$f^*(s) = \frac{\prod_{j=1}^{p} \Gamma(a_j + s)}{\prod_{j=1}^{q} \Gamma(b_j + s)} , \qquad \mathrm{Re} s > 0 .$$

Therefore (2) becomes

$$\int_0^\infty \cdots \int_0^\infty G^{p\,0}_{q\,p} \left(\max\{x_1, \cdots, x_n\} \mid {b_1, \cdots, b_q \atop a_1, \cdots, a_p} \right) \prod_{i=1}^{n} x_i^{s_i - 1} dx_1 \cdots dx_n = \tag{4}$$

$$= \frac{\prod_{j=1}^{p} \Gamma(a_j + s_1 + \cdots + s_n)}{\prod_{j=1}^{q} Gamma(b_j + s_1 + \cdots + s_n)} \frac{(s_1 + \cdots + s_n)}{s_1 \cdots s_n} , \qquad \mathrm{Re} s_1, ; \cdots, \mathrm{Re} s_n > 0 .$$

The representation for the generalized Lauricella function can now be obtained as follows. Let $p_i \leq q_i$, $i = 1, 2, \cdots, n$, and either $p = q$ and $\mathrm{Re} \sum_{j=1}^{p} a_j < \mathrm{Re} \sum_{j-1}^{q} b_j$ or $p > q$. Then

$$F^{p\cdot p_1, \cdots, p_n}_{q \cdot q_1; \cdots ; q_n} \left({a_p ; c^1_{p_1}; \cdots ; c^n_{p_n} ; \atop b_q ; d^1_{q_1}; \cdots ; d^n_{q_n} ;} x_1, \cdots, x_n \right)$$

$$= \frac{1}{n} \sum_{m_1, \cdots, m_n = 0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{m_1 + \cdots + m_n} (n)_{m_1 + \cdots + m_n} (n + m_1 + \cdots + m_n)}{\prod_{j=1}^{q} (b_j)_{m_1 + \cdots + m_n} (n+1)_{m_1 + \cdots + m_n} (m_1 + 1) \cdots (m_n + 1)} \cdot$$

$$\cdot \prod_{i=1}^{n} \left\{ \frac{\prod_{j=1}^{p_i} (c^i_j)_{m_i} (2)_{m_i} x_i^{m_i}}{\prod_{j=1}^{q_i} (d^i_j)_{m_i} (1)_{m_i} m_i!} \right\}$$

$$= \frac{1}{n} \frac{\Gamma(b_1) \cdots \Gamma(b_q)}{\Gamma(a_1) \cdots \Gamma(a_p)} \int_0^\infty \cdots \int_0^\infty G^{p+1,0}_{q+1,p+1} \left(\max\{t_1, \cdots, t_n\} \mid {b_1 - n, \cdots, b_q - n, 1 \atop a_1 - n, \cdots, a_p - n, 0} \right) \cdot$$

$$\cdot \prod_{i=1}^{n} \left\{ \sum_{m_i = 0}^{\infty} \frac{\prod_{j=1}^{p_i} (c^i_j)_{m_i} (2)_{m_i} (x_i t_i)^{m_i}}{\prod_{j=1}^{q_i} (d^i_j)_{m_i} (1)_{m_i} m_i!} \right\} dt_1 \cdots dt_n .$$

Thus we have proved

$$F_{q\, q_1, \quad ,q_n}^{p\, p_1, \quad ,p_n}\left(\begin{matrix} a_p; c_{p_1}^1; \cdots; c_{p_n}^n; \\ b_q; d_{q_1}^1; \cdots; d_{q_n}^n; \end{matrix} x_1, \cdots, x_n\right) = \tag{5}$$

$$= \frac{1}{n} \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)} \cdot \subset n t_0^{\infty} \cdots \int_0^{\infty} G_{q+1,p+1}^{p+1,0}\left(\max\{t_1,\cdots,t_n\} \Big|_{a_1-n, \quad ,a_p-n,0}^{b_1-n, \quad ,b_q-n,1}\right) \cdot$$

$$\cdot \prod_{i=1}^{n} {}_{p_i+1}F_{q_i+1}\left(\begin{matrix} c_1^i, \cdots, c_{p_i}^i, 2; \\ d_1^i, \cdots, d_{q_i}^i, 1; \end{matrix} x_i t_i\right) dt_1 \cdots dt_n \,.$$

Formula (5) is valid when $\mathrm{Re} a_j > n$; $j = 1, 2, \cdots, p$; $p_i \le q_i$, $i = 1, 2, \cdots, n$; $p \ge q$, $1 + q + q_i - p - p_i \ge 0$; $i = 1, 2, \cdots, n$. (The restriction $\mathrm{Re} \sum_{j=1}^{q} b_j + 1 > \mathrm{Re} \sum_{j=1}^{p} a_j$ is needed when $p = q$.)

4. INTEGRAL REPRESENTATIONS OF LAURICELLA HYPERGEOMETRIC FUNCTIONS.

Derivations similar to those in the previous section lead to representations for 3 of the Lauricella functions. (Since in (5) $p \ge q$ we do not get $F_B^{(n)}$.) We introduce the operator

$$\mathcal{D}_i = \left(\frac{a-1}{n} + x_i \frac{d}{d(x_i)}\right) \,.$$

(a) Using the formula (2) we get

$$\int_0^{\infty} \cdots \int_0^{\infty} \exp(-\max\{x_1, \cdots, x_n\}) \prod_{i=1}^{n} x_i^{\frac{a-1}{n} + m_i - 1} dx_1 \cdots dx_n = \tag{6}$$

$$= \frac{\Gamma(a + m_1 + \cdots + m_n)}{\left(\frac{a-1}{n} + m_1\right) \cdots \left(\frac{a-1}{n} + m_n\right)} \,, \qquad \mathrm{Re} a > 1 \,.$$

Consequently, for the Lauricella function $F_A^{(n)}$ we have

$$F_A^{(n)}(a; b_1, \cdots, b_n; c_1, \cdots, c_n; x_1, \cdots, x_n) =$$

$$= \sum_{m_1, \quad ,m_n=0}^{\infty} \frac{(a)_{m_1 + \quad + m_n} (b_1)_{m_1} \cdots (b_n)_{m_n}}{(c_1)_{m_1} \cdots (c_n)_{m_n} m_1! \cdots m_n!} x_1^{m_1} \cdots x_n^{m_n} =$$

$$= \sum_{m_1, \quad ,m_n=0}^{\infty} \frac{1}{\Gamma(a)} \int_0^{\infty} \cdots \int_0^{\infty} \prod_{i=1}^{n} t_i^{\frac{a-1}{n} + m_i - 1} \exp(-\max\{t_1, \cdots, t_n\})$$

$$\cdot \prod_{i=1}^{n} \frac{\left(\frac{a-1}{n} + m_i\right) (b_i)_{m_i} x_i^{m_i}}{(c_i)_{m_i} m_i!} dt_1 \cdots dt_n =$$

$$= \frac{1}{\Gamma(a)} \int_0^{\infty} \cdots \int_0^{\infty} (t_1 \cdots t_n)^{\frac{a-1}{n} - 1} \exp(-\max\{t_1, \cdots, t_n\})$$

$$\cdot \prod_{i=1}^{n} \left\{ \sum_{m_i=0}^{\infty} \left(\frac{a-1}{n} + m_i\right) \frac{(b_i)_{m_i} (t_i x_i)^{m_i}}{(c_i)_{m_i} m_i!} \right\} dt_1 \cdots dt_n \,,$$

Consequently

$$F_A^{(n)}(a; b_1, \cdots, b_n; c_1, \cdots, c_n; x_1, \cdots, x_n) = \tag{7}$$

$$= \frac{1}{\Gamma(a)} \prod_{i=1}^{n} \mathcal{D}_i \int_0^{\infty} \cdots \int_0^{\infty} \prod_{i=1}^{n} t_i^{\frac{a-1}{n} - 1} \exp(-\max\{t_1, \cdots, t_n\}) \cdot \prod_{i=1}^{n} {}_1F_1(b_i, c_i; x_i t_i) \, dt_1 \cdots dt_n \,,$$

$$\mathrm{Re} a > 1 \,.$$

(b) From (2) we have

$$\int_0^{1} \cdots \int_0^{1} (1 - \max\{x_1, \cdots, x_n\})^{b-a} \prod_{i=1}^{n} x_i^{\frac{a-1}{n} + m_i - 1} dx_1 \cdots dx_n = \tag{8}$$

$$= \frac{\Gamma(a + m_1 + \cdots + m_n) \Gamma(b - a + 1)}{\Gamma(b + m_1 + \cdots + m_n) \left(\frac{a-1}{n} + m_1\right) \cdots \left(\frac{a-1}{n} + m_n\right)} \,, \qquad 1 + \mathrm{Re} b > \mathrm{Re} a > 1 \,.$$

Therefore for the Lauricella function $F_D^{(n)}$ we get

$$F_D^{(n)}(a;b_1,\cdots,b_n;c;x_1,\cdots,x_n)=$$

$$=\sum_{m_1,\ ,m_n=0}^{\infty}\frac{(a)_{m_1+\ +m_n}(b_1)_{m_1}\ (b_n)_{m_n}x_1^{m_1}\cdots x_n^{m_n}}{(c)_{m_1+\ +m_n}m_1!\cdots m_n!}=$$

$$=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a+1)}\int_0^1\cdots\int_0^1\prod_{i=1}^n t_i^{\frac{a-1}{n}-1}(1-\max\{t_1,\cdots,t_n\})^{c-a}$$

$$\prod_{i=1}^n\sum_{m_i=0}^{\infty}\left(\frac{a-1}{n}+m_i\right)\cdot\frac{(b_i)_{m_i}(x_it_i)^{m_i}}{m_i!}dt_1\cdots dt_n\ .$$

Hence

$$F_D^{(n)}(a;b_1,\cdots,b_n;c;x_1,\cdots,x_n) \tag{9}$$

$$=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a+1)}\prod_{i=1}^n\mathcal{D}_i\int_0^1\cdots\int_0^1\prod_{i=1}^n t_i^{\frac{a-1}{n}-1}(1-\max\{t_1,\cdots,t_n\})^{c-a}\prod_{i=1}^n(1-x_it_i)^{-b_i}dt_1\cdots dt_n,$$

$$1+\mathrm{Re}c>\mathrm{Re}a>1\ .$$

(c) Using the formula (2) again, we have

$$\int_0^\infty\cdots\int_0^\infty(\max\{x_1,\cdots,x_n\})^{\frac{b-a+1}{2}}K_{b-a+1}\left(2(\max\{t_1\cdots t_n\})^{\frac12}\right)\prod_{i=1}^n(x_i)^{\frac{a-1}{n}+m_i-1}dx_1\cdots dx_n= \tag{10}$$

$$=\frac{1}{2}\frac{\Gamma(a+m_1+\cdots+m_n)\Gamma(b+m_1+\cdots+m_n)}{(\frac{a-1}{n}+m_1)\cdots(\frac{a-1}{n}+m_n)}\ ,\quad \mathrm{Re}a>1\ ,\quad \mathrm{Re}b>-0\ .$$

Consequently, for the Lauricella hypergeometric function $F_C^{(n)}$, by a similar development we have the result

$$F_C^{(n)}(a;b;c_1,\cdots,c_n;x_1,\cdots,x_n)= \tag{11}$$

$$=\frac{2}{\Gamma(a)\Gamma(b)}\prod_{i=1}^n\mathcal{D}_i\int_0^\infty\cdots\int_0^\infty\prod_{i=1}^n t_i^{\frac{a-1}{n}-1}(\max\{t_1,\cdots,t_n\})^{\frac{b-a+1}{2}}K_{b-a+1}\left(2(\max\{t_1\cdots t_n\})^{\frac12}\right)$$

$$\prod_{i=1}^n{}_0F_1(c_i;x_it_i)\,dt_1\cdots dt_n,$$

$$\mathrm{Re}a>1,\quad \mathrm{Re}b>-1\ .$$

For Appell functions these read

$$F_1(a;b_1,b_2;c;x_1,x_2)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a+1)} \tag{12}$$

$$\mathcal{D}_1\mathcal{D}_2\int_0^1\int_0^1(1-\max\{t_1,t_2\})^{c-a}(t_1t_2)^{(a-3)/2}\left\{(1-x_1t_1)^{-b_1}\right\}\left\{(1-x_2t_2)^{-b_2}\right\}dt_1dt_2\ ,$$

$$1+\mathrm{Re}c>\mathrm{Re}a>1\ ;$$

$$F_2(a;b_1,b_2;c_1,c_2;x_1,x_2)= \tag{13}$$

$$=\frac{1}{\Gamma(a)}\mathcal{D}_1\mathcal{D}_2\int_0^\infty\int_0^\infty(t_1t_2)^{(a-3)/2}\exp(-\max\{t_1,t_2\})$$

$$\{{}_1F_1(b_1;c_1;x_1t_1)\}\{{}_1F_1(b_2;c_2;x_2t_2)\}\,dt_1dt_2\ ,$$

$$\mathrm{Re}a>1\ ;$$

$$F_4(a;b;c_1,c_2;x_1,x_2) = \tag{14}$$

$$= \frac{2\Gamma(c_1)\Gamma(c_2)}{\Gamma(a)\Gamma(b)}\mathcal{D}_1\mathcal{D}_2\int_0^\infty\int_0^\infty (\max\{t_1,t_2\})^{(b-a+1)/2}(t_1t_2)^{(a-3)/2}K_{b-a+1}\left(2(\max\{t_1,t_2\})^{1/2}\right)$$

$$\cdot\left\{(x_1t_1)^{(1-c_1)/2}I_{c_1-1}\left(2(x_1t_1)^{1/2}\right)\right\}\left\{(x_2t_2)^{(1-c_2)/2}I_{c_2-1}\left(2(x_2t_2)^{1/2}\right)\right\}dt_1dt_2\,,$$

$$Re a > 1\,,\quad Re b > 0\,.$$

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