Internat. J. Math. & Math. Sci.
VOL. 16 NO. 1 (1993) 149-154

A PROOF OF COMPLETENESS OF THE GREEN-LAMÉ TYPE SOLUTION IN THERMOELASTICITY

D.S. CHANDRASEKHARAIAH

Department of Mathematics
University of Central Florida
Orlando, Florida 32816

(Received December 26, 1991)

ABSTRACT. A proof of completeness of the Green-Lamé type solution for the unified governing field equations of conventional and generalized thermoelasticity theories is given.

KEY WORDS AND PHRASES. Thermoelasticity, Generalized Thermoelasticity, Green-Lamé solution, completeness of solution.
1980 AMS SUBJECT CLASSIFICATION CODE. 73U.

1. INTRODUCTION

In [1], the author presented three complete solutions for the following system of coupled partial differential equations which may be interpreted as a unified system of governing field equations of the conventional and generalized models of the linear thermoelasticity theory of homogeneous and isotropic materials:

$$\left.\begin{aligned} &\left(c^2\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\underset{\sim}{u} + \left(1-c^2\right)\nabla\, div\, \underset{\sim}{u} - \left(1+\alpha\frac{\partial}{\partial t}\right)\nabla\theta + \underset{\sim}{F} = \underset{\sim}{0} \\ &\left(\nabla^2 - \frac{\partial}{\partial t} - \beta\frac{\partial^2}{\partial t^2}\right)\theta - \left(1+\gamma\frac{\partial}{\partial t}\right)\left[\varepsilon\frac{\partial}{\partial t}(div\, \underset{\sim}{u}) - h\right] = 0 \end{aligned}\right\} \quad (1.1\text{ a,b})$$

The notation employed in these equations and those to follow are as explained in [1].

One of the three solutions of the system (1.1) presented in [1] is analogous to the Green-Lamé solution in classical elastodynamics [2]; this solution is described by the following relations:

$$\underset{\sim}{u} = \left(1+\alpha\frac{\partial}{\partial t}\right)(\nabla\phi + curl\, \underset{\sim}{\psi}) \quad (1.2)$$

$$\theta = D_1\phi - f \quad (1.3)$$

$$D_5\phi = D_3 f - \left(1+\gamma\frac{\partial}{\partial t}\right)h \quad (1.4)$$

$$D_2\underset{\sim}{\psi} = \underset{\sim}{g} \quad (1.5)$$

$$\underset{\sim}{F} = -\left(1+\alpha\frac{\partial}{\partial t}\right)(\nabla f + curl\, \underset{\sim}{g}) \quad (1.6)$$

That is, if the known function $\underset{\sim}{F}$ is represented by the relation (1.6) (by virtue of the Helmholtz resolution of a vector field), then a solution $\{\underset{\sim}{u}, \theta\}$ for the system (1.1) is given by the representations (1.2) and (1.3) where ϕ and $\underset{\sim}{\psi}$ are arbitrary scalar and vector functions (respectively) obeying the partial differential equations (1.4) and (1.5). Here D_1, D_2, D_3 and D_5 are partial differential operators defined by [1]:

$$D_1 = \nabla^2 - \frac{\partial^2}{\partial t^2} \tag{1.7}$$

$$D_2 = c^2\nabla^2 - \frac{\partial^2}{\partial t^2} = D_1 - \left(1 - c^2\right)\nabla^2 \tag{1.8}$$

$$D_3 = \nabla^2 - \frac{\partial}{\partial t} - \beta\frac{\partial^2}{\partial t^2} \tag{1.9}$$

$$D_5 = D_3 D_1 - \varepsilon\nabla^2\frac{\partial}{\partial t}\left(1 + \alpha\frac{\partial}{\partial t}\right)\left(1 + \gamma\frac{\partial}{\partial t}\right) \tag{1.10}$$

It was also shown in [1] that the solution described above is complete in the sense that if the known function $\underset{\sim}{F}$ is represented as in (1.6), then every solution $\{\underset{\sim}{u}, \theta\}$ of the system (1.1) admits a representation given by the relations (1.2) and (1.3) with ϕ and $\underset{\sim}{\psi}$ obeying the equations (1.4) and (1.5).

The proof of completeness suggested in [1] was an extension of the proof given in [2] in the context of classical elastodynamics. This proof makes the hypothesis that in the representation (1.6) for $\underset{\sim}{F}$ the function $\underset{\sim}{g}$ is divergence-free (that is, $div\ \underset{\sim}{g} = 0$) and infers that $\underset{\sim}{\psi}$ also has to be divergence-free.

The object of the present Note is to give a proof of the completeness of the Green-Lamé type solution that does not make the hypothesis that $div\ \underset{\sim}{g} = 0$ and consequently does not infer that $div\ \underset{\sim}{\psi} = 0$. This proof is motivated by the work of Long [3] in classical elastodynamics and is analogous to that given in [4] in the context of the theory of elastic materials with voids.

2. PROOF OF COMPLETENESS

Consider <u>any</u> solution $\{\underset{\sim}{u}, \theta\}$ of the system (1.1). By virtue of the Helmholtz representation of a vector field, $\underset{\sim}{u}$ may be expressed as

$$\underset{\sim}{u} = \left(1 + \alpha\frac{\partial}{\partial t}\right)(\nabla p + curl\ \underset{\sim}{q}) \tag{2.1}$$

for some scalar field p and a vector field $\underset{\sim}{q}$.

Substituting for $\underset{\sim}{u}$ from (2.1) into equation (1.1a), we get the equation

$$\left(1 + \alpha\frac{\partial}{\partial t}\right)\left[\nabla\{D_1 p - (\theta + f)\} + curl\{D_2\underset{\sim}{q} - \underset{\sim}{g}\}\right] \tag{2.2}$$

Here, we have made use of the representation (1.6) for $\underset{\sim}{F}$ and the relations (1.7) and (1.8).

For $\alpha = 0$, equation (2.2) gives

$$\nabla\{D_1 p - (\theta + f)\} = curl\{\underset{\sim}{g} - D_2\underset{\sim}{q}\} \tag{2.3}$$

For $\alpha \neq 0$, equation (2.2) yields equation (2.3) provided

$$\left[\nabla\{D_1 p - (\theta + f)\} + curl\{D_2\underset{\sim}{q} - \underset{\sim}{g}\}\right]_{t=0} = \underset{\sim}{0}.$$

This condition may be taken to be valid when $\underset{\sim}{u}$ and θ obey homogeneous initial conditions.

Taking the divergence of both sides of (2.3) and noting that $div\ \nabla = \nabla^2$ and $div\ curl$ is the zero operator, we get the equation

$$\nabla^2\{D_1 p-(\theta+f)\}=0. \tag{2.4}$$

This equation implies that [4, Appendix]

$$p=\phi+\phi_0 \tag{2.5}$$

where

$$D_1\phi=\theta+f \tag{2.6}$$

$$\nabla^2\phi_0=0. \tag{2.7}$$

Taking the *curl curl* of both sides of (2.3) and noting that *curl* ∇ is the zero operator and $curl\ curl=\nabla div-\nabla^2$, we obtain the equation

$$\nabla^2 curl(D_2\underset{\sim}{q}-\underset{\sim}{g})=\underset{\sim}{0}. \tag{2.8}$$

This equation implies that [4, Appendix]

$$\underset{\sim}{q}=\underset{\sim}{\psi}_0+\underset{\sim}{\psi}_1 \tag{2.9}$$

where

$$\nabla^2(curl\ \underset{\sim}{\psi}_0)=\underset{\sim}{0} \tag{2.10}$$

$$D_2\underset{\sim}{\psi}_1=\underset{\sim}{g} \tag{2.11}$$

Substituting for p and $\underset{\sim}{q}$ from (2.5) and (2.9) into the expression (2.3) and using (2.6) and (2.11) we obtain the relation

$$\nabla(D_1\phi_0)+curl(D_2\underset{\sim}{\psi}_0)=\underset{\sim}{0}.$$

Using the relations (1.7), (1.8), (2.7) and (2.10), this yields

$$\frac{\partial^2}{\partial t^2}\{\nabla\phi_0+curl\ \underset{\sim}{\psi}_0\}=\underset{\sim}{0}$$

from which it follows that

$$\nabla\phi_0+curl\ \underset{\sim}{\psi}_0=t\ \underset{\sim}{\psi}_2+\underset{\sim}{\psi}_3 \tag{2.12}$$

where $\underset{\sim}{\psi}_2$ and $\underset{\sim}{\psi}_3$ are independent of t.

Taking the divergence of (2.12) and using (2.7), we get

$$t\ div\ \underset{\sim}{\psi}_2+div\ \underset{\sim}{\psi}_3=0.$$

Since this holds for any t, we should have $div\ \underset{\sim}{\psi}_2=0$ and $div\ \underset{\sim}{\psi}_3=0$ from which it follows that

$$\underset{\sim}{\psi}_2=curl\ \underset{\sim}{\xi}_2,\quad \underset{\sim}{\psi}_3=curl\ \underset{\sim}{\xi}_3 \tag{2.13}$$

for some $\underset{\sim}{\xi}_2, \underset{\sim}{\xi}_3$.

Taking the Laplacian of (2.12) and using (2.7) and (2.10), we get

$$t\ \nabla^2\underset{\sim}{\psi}_2+\nabla^2\underset{\sim}{\psi}_3=\underset{\sim}{0}$$

which on using (2.13) yields

$$t\ curl(\nabla^2\underset{\sim}{\xi}_2)+curl(\nabla^2\underset{\sim}{\xi}_3)=\underset{\sim}{0}.$$

Since this holds for any t, we should have $curl\left(\nabla^2 \underset{\sim}{\xi}_2\right) = \underset{\sim}{0}$ and $curl\left(\nabla^2 \underset{\sim}{\xi}_3\right) = \underset{\sim}{0}$ from which it follows that

$$\nabla^2 \underset{\sim}{\xi}_2 = \nabla \phi_2, \quad \nabla^2 \underset{\sim}{\xi}_3 = \nabla \phi_3 \tag{2.14}$$

for some ϕ_2 and ϕ_3.

We now define the function $\underset{\sim}{\psi} = \underset{\sim}{\psi}(P, t)$ by

$$\underset{\sim}{\psi} = \underset{\sim}{\psi}_1 + (t\,\underset{\sim}{\xi}_2 + \underset{\sim}{\xi}_3) + \frac{1}{4\pi} \nabla \int_D \frac{\Phi(Q, t - R/c)}{R} dV \tag{2.15}$$

where

$$\Phi = t\,\phi_2 + \phi_3 \tag{2.16}$$

and R is the distance from the field point P to a point Q, the integration (over D) being w.r.t. Q.

From (2.15) we get

$$curl\ \underset{\sim}{\psi} = curl\left(\underset{\sim}{\psi}_1 + t\,\underset{\sim}{\xi}_2 + \underset{\sim}{\xi}_3\right). \tag{2.17}$$

Substituting for p and $\underset{\sim}{q}$ from (2.5) and (2.9) in the right-hand side of (2.1) and using (2.12), (2.13) and (2.17), we obtain

$$\underset{\sim}{u} = \left(1 + \alpha \frac{\partial}{\partial t}\right)\left(\nabla \phi + curl\ \underset{\sim}{\psi}\right).$$

This is the desired representation (1.2) for $\underset{\sim}{u}$. The desired representation (1.3) for θ is given by (2.6).

Substituting for $\underset{\sim}{u}$ and θ from (1.2) and (1.3) into equation (1.1b) and using (1.9) and (1.10), we obtain the equation

$$D_5 \phi - D_3 f + \left(1 + \gamma \frac{\partial}{\partial t}\right) h = 0.$$

This is precisely the desired governing equation (1.4) for ϕ.

With the aid of the identity [1]

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial}{\partial t^2}\right) \int_D \frac{\Phi(Q, t - R/c)}{R} dV = -4\pi \Phi$$

and the relations (1.8), (2.14) and (2.16), expression (2.15) yields $D_2 \underset{\sim}{\psi} = D_2 \underset{\sim}{\psi}_1$. Using the relation (2.11), we now find that $\underset{\sim}{\psi}$ obeys the equation $D_2 \underset{\sim}{\psi} = \underset{\sim}{g}$, which is the desired governing equation (1.5) for $\underset{\sim}{\psi}$.

Thus, we have shown that, given any solution $\{\underset{\sim}{u}, \theta\}$ for the system (1.1), one can construct functions ϕ and $\underset{\sim}{\psi}$ such that $\underset{\sim}{u}$ and θ can be represented by the relations (1.2) and (1.3) with ϕ and $\underset{\sim}{\psi}$ obeying the equations (1.4) and (1.5).

This completes the proof of completeness of the Green-Lamé type solution for the system (1.1). Note that no where in the proof it has been assumed that $div\ \underset{\sim}{g} = 0$ and inferred that $div\ \underset{\sim}{\psi}$ has to be zero.

ACKNOWLEDGEMENT. This work is supported by the U. S. Government Fulbright Grant #15068 under the Indo-American Fellowship Program. The author is thankful to Bangalore University, Bangalore and the University Grants Commission, New Delhi for nominating him for the Fellowship and the Indo-U.S. subcommission on education and culture for awarding the Fellowship. His thanks are also due to Professor Lokenath Debnath for the facilities.

REFERENCES

1. CHANDRASEKHARAIAH, D. S.: Complete solutions of a coupled system of partial differential equations arising in Thermoelasticity, Quart. Appl. Math. XLV (1987) 471-480.

2. GURTIN, M. E.: The linear theory of elasticity, Encyclopedia of Physics, Vol. VI a/2, Springer-Verlag, New York (1972), p. 234.

3. LONG, C. F.: On the completeness of the Lamé potentials, Acta Mechanica 3 (1967) 371-375.

4. CHANDRASEKHARAIAH, D. S.: Complete solutions in the theory of elastic materials with voids, Quart. J. Mech. Appl. Math. 37 (1987) 401-416.

Permanent address of the author:

Department of Mathematics, Bangalore University, Central College Campus, Bangalore 560 001, India.