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ON CERTAIN CLASSES OF p-VALENT ANALYTIC FUNCTIONS

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ABSTRACT. The objective of the present paper is to introduce a certain general class $P(p,\alpha,\beta)(p \in N = \{1,2,3,...\}, 0 \leq \alpha < p$ and $\beta \geq 0)$ of p-valent analytic functions in the open unit disk U and we prove that if $f \in P(p,\alpha,\beta)$ then $J_{p,c}(f)$, defined by

$$J_{p,c}(f) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)dt \qquad (c \in N),$$

belongs to $P(p,\alpha,\beta)$. We also investigate inclusion properties of the class $P(p,\alpha,\beta)$. Furthermore, we examine some properties for a class $T_p(\alpha,\beta)$ of analytic functions with negative coefficients.

KEY WORDS AND PHRASES. p-valent analytic function, Hadamard product, integral operator, multiplier transformation, p-valently convex of order δ.
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1. INTRODUCTION.

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \ (p \in N = \{1,2,3,...\}) \tag{1.1}$$

which are analytic in the unit disk $U = \{z: |z| < 1\}$. We also denote by S_p the subclass of A_p consisting of functions which are p-valent in U.

A function $f \in A_p$ is said to be in the class $P(p,\alpha)$ $(0 \leq \alpha < p)$ if and only if it satisfies the inequality

$$Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha \qquad (0 \leq \alpha < p, z \in U). \tag{1.2}$$

The classes $P(1,0)$ and $P(p,0)$ were investigated by MacGregor [7] and Umezawa [11], respectively. In fact, the class $P(p,\alpha)$ is a subclass of the class S_p [11].

Let f and g be in the class A_p, with $f(z)$ given by (1.1), and $g(z)$ defined by

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{n+p} z^{n+p}. \tag{1.3}$$

The convolution or Hadamard product of f and g is defined by

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} b_{n+p} z^{n+p}. \tag{1.4}$$

For a function $f \in A_p$ given by (1.1), Reddy and Padmanabhan [10] defined the integral operator $J_{p,c}$ $(p, c \in N)$ by

$$\begin{aligned} J_{p,c}(f) &= \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)dt \\ &= z^p + \sum_{n=1}^{\infty} \frac{c+p}{c+p+n} a_{n+p} z^{n+p}. \end{aligned} \tag{1.5}$$

The operator $J_{1,c}$ was introduced by Bernardi [2]. In particular, the operator $J_{1,1}$ were studied by Libera [5] and Livingston [6].

Clearly, (1.5) yields

$$f \in A_p \Rightarrow J_{p,c} \in A_p \tag{1.6}$$

Thus, by applying the operator $J_{p,c}$ successively, we can obtain

$$J^n_{p,c}(f) = \begin{cases} J_{p,c}(J^{n-1}_{p,c}(f)) & (n \in N), \\ f(z) & (n = 0). \end{cases} \tag{1.7}$$

We now recall the following definition of a multiplier transformation (or fractional integral and fractional derivative).

DEFINITION 1([3]). Let the function

$$\phi(z) = \sum_{n=0}^{\infty} c_{n+p} z^{n+p} \tag{1.8}$$

be analytic in U and let λ be a real number. Then the multiplier transformation $I^\lambda \phi$ is defined by

$$I^\lambda \phi(z) = \sum_{n=0}^{\infty} (n+p+1)^{-\lambda} c_{n+p} z^{n+p} \qquad (z \in U). \tag{1.9}$$

The function $I^\lambda \phi$ is clearly analytic in U. It may be regarded as a fractional integral (for $\lambda > 0$) or fractional derivative (for $\lambda < 0$) of ϕ. Furthermore, in terms of the Gamma function, we have

$$I^\lambda \phi(z) = \frac{1}{\Gamma(z)}\int_0^1 (log\tfrac{1}{t})^{\lambda-1}\phi(zt)dt \qquad (\lambda > 0). \tag{1.10}$$

DEFINITION 2. The fractional derivative $D^\lambda \phi$ of order $\lambda \geq 0$, for an analytic function ϕ given by (1.8), is defined by

$$D^\lambda \phi(z) = I^{-\lambda}\phi(z) = \sum_{n=0}^{\infty} (n+p+1)^\lambda c_{n+p} z^{n+p} \qquad (\lambda \geq 0, z \in U). \tag{1.11}$$

Making use of Definition 2, we now introduce an interesting generalization of the class $P(p,\alpha)$ of functions in A_p which satisfy the inequality (1.2).

DEFINITION 3. A function $f \in A_p$ is said to be in the class $P(p,\alpha,\beta)$ if and only if

$$(p+1)^{-\beta} D^\beta f \in P(p,\alpha) \qquad (0 \leq \alpha < p, \beta \geq 0)$$

Observe that $P(p,\alpha,0) = P(p,\alpha)$. Furthermore, since $f \in A_p$, it follows from (1.1) and (1.9) that

$$(p+1)^{-\beta} D^{\beta}f(z) = z^p + \sum_{n=1}^{\infty}\left(\frac{n+p+1}{p+1}\right)^{\beta} a_{n+p}\, z^{n+p}, \tag{1.12}$$

which shows that $(p+1)^{-\beta} D^{\beta}f \in A_p$ if $f \in A_p$. In particular, the class $P(1,\alpha,\beta)$ was introduced by Kim, Lee, and Srivastava [4].

2. SOME INCLUSION PROPERTIES.

In our present investigation of the general class $P(p,\alpha,\beta)$ $(0 \le \alpha < p, \beta \ge 0)$, we need the following lemma.

LEMMA 2.1([1]). Let $M(z)$ and $N(z)$ be analytic in U, $N(z)$ map U onto a many sheeted starlike region of order γ $(0 \le \gamma < p)$ and

$$M(0) = N(0) = 0,\ \frac{M'(0)}{N'(0)} = p, \qquad Re\left(\frac{M'(z)}{N'(z)}\right) > \gamma.$$

Then we have

$$Re\left(\frac{M(z)}{N(z)}\right) > \gamma \qquad (0 \le \gamma < p, p \ge 1).$$

By using Lemma 2.1, we can prove

THEOREM 2.1. Let the function $f(z)$ be in the class $P(p,\alpha,\beta)$. Then $J_{p,c}(f)$ defined by (1.5) is also in the class $P(p,\alpha,\beta)$.

PROOF. A simple calculation shows that

$$\frac{\frac{d}{dz}D^{\beta}(J_{p,c}(f))}{z^{p-1}} = \frac{c+p}{z^{c+p}}\int_0^z t^c\left\{\frac{d}{dt}D^{\beta}f(t)\right\}dt \tag{2.1}$$

where the operators $J_{p,c}$ $(c \in N)$ and D^{λ} $(\lambda \ge 0)$ are defined by (1.5) and (1.11), respectively. In view of (2.1), we get

$$M(z) = \frac{c+p}{(p+1)^{\beta}}\int_0^z t^c\left\{\frac{d}{dt}D^{\beta}f(t)\right\}dt \text{ and } N(z) = z^{p+c}, \tag{2.2}$$

so that

$$Re\left\{\frac{M'(z)}{N'(z)}\right\} = Re\left\{\frac{(p+1)^{-\beta}\frac{d}{dz}D^{\beta}f(z)}{z^{p-1}}\right\}. \tag{2.3}$$

Since, by hypothesis, $f \in P(p,\alpha,\beta)$, the second member of (2.3) is greater than α, and hence

$$Re\left\{\frac{M'(z)}{N'(z)}\right\} > \alpha \qquad (o \le \alpha < p). \tag{2.4}$$

Thus, by Lemma 2.1, we have

$$Re\left\{\frac{M(z)}{N(z)}\right\} = Re\left\{\frac{(p+1)^{-b}\frac{d}{dz}D^{\beta}(J_{p,c}(f))}{z^{p-1}}\right\} > \alpha \qquad (o \le \alpha < p, \beta \ge 0), \tag{2.5}$$

which completes the proof of Theorem 2.1.

Let $f \in A_p$ be given by (1.1). Suppose also that

$$F_m(f) = J_{p,c_1}\left(\cdots\left(J_{p,c_m}(f)\right)\right)$$

$$= z^p + \sum_{n=1}^{\infty} \frac{(c_1+p)\dots(c_m+p)}{(c_1+p+n)\dots(c_m+p+n)} a_{n+p}\, z^{n+p}\ (c_j \in N(j=1,2,\dots m), m \in N). \qquad (2.6)$$

Then, by Theorem 2.1, we have

COROLLARY 2.1. Let the function $f(z)$ be in the class $P(p,\alpha,\beta)$. Then the function $F_m(f)$ defined by (2.6) is also in the class $P(p,\alpha,\beta)$.

The next inclusion property of the class $P(p,\alpha,\beta)$, contained in Theorem 2.2 below, would involve the operator $J^{\lambda}_{p,1}(\lambda > 0)$ defined by

$$J^{\lambda}_{p,1}(f) = (1+p)^{\lambda} I^{\lambda} f(z) \qquad (\lambda > 0, f \in A_p). \qquad (2.7)$$

For $\lambda = m \in N$, we have

$$J^{m}_{p,1}(f) = (1+p)^{m} I^{m} f(z)$$

$$= \frac{(1+p)^m}{(m-1)!}\int_0^1 (log\tfrac{1}{t})^{m-1} f(t)dt. \qquad (2.8)$$

Clearly, we have

$$f \in A_p \Rightarrow J^{\lambda}_{p,1}(f) \in A_p \qquad (\lambda > 0). \qquad (2.9)$$

THEOREM 2.2. Let the function $f(z)$ be in the class $P(p,\alpha,\beta)$. Then the function $J^{\lambda}_{p,1}(\lambda > 0)$ defined by (2.7) is also in the class $P(p,\alpha,\beta)$.

PROOF. Making use of (1.9) and (1.11), the definition (2.7) yields

$$(p+1)^{-\beta} D^{\beta}(J^{\lambda}_{p,1}(f)) = J^{\lambda}_{p,1}((p+1)^{-\beta} D^{\beta} f) \qquad (\beta \geq 0, \lambda > 0, f \in A_p) \qquad (2.10)$$

Therefore, setting

$$g(z) = (p+1)^{-\beta} D^{\beta} f \text{ and } G(z) = J^{\lambda}_{p,1}(g), \qquad (2.11)$$

we must show that

$$Re\left\{\frac{G'(z)}{z^{p-1}}\right\} > \alpha \qquad (0 \leq \alpha < p) \qquad (2.12)$$

whenever $f \in P(p,\alpha,\beta)$.

From the integral representation in (1.10), we obtain

$$G'(z) = \frac{(p+1)^{\lambda}}{\Gamma(z)}\int_0^1 (log\tfrac{1}{t})^{\lambda-1} t g'(zt)dt \qquad (\lambda > 0), \qquad (2.13)$$

so that

$$Re\left\{\frac{G'(z)}{z^{p-1}}\right\} = \frac{(p+1)^{\lambda}}{\Gamma(\lambda)}\int_0^1 (log\tfrac{1}{t})^{\lambda-1} t^p Re\left\{\frac{g'(zt)}{(zt)^{p-1}}\right\}dt \qquad (\lambda > 0), \qquad (2.14)$$

Since $f \in P(p,\alpha,\beta)$, we have

$$Re\left\{\frac{g'(zt)}{(zt)^{p-1}}\right\} > \alpha \qquad (0 \leq \alpha < p, 0 \leq t \leq 1), \qquad (2.15)$$

and hence (2.14) yields

$$Re\left\{\frac{G'(z)}{z^{p-1}}\right\} = \frac{(p+1)^{\lambda}}{\Gamma(\lambda)}\alpha\int_0^1 (log\tfrac{1}{t})^{\lambda-1}t^p dt = \alpha \qquad (0 \le \alpha < p, \lambda > 0), \tag{2.16}$$

which completes the proof of Theorem 2.2.

COROLLARY 2.2. If $0 \le \alpha < p$ and $0 \le \beta < \gamma$, then $P(p,\alpha,\gamma) \subset P(p,\alpha,\beta)$.

PROOF. Setting $\lambda = \gamma - \beta > 0$ in Theorem 2.2, we observe that

$$\begin{aligned} f \in P(p,\alpha,\gamma) &\Rightarrow J_{p,1}^{\gamma-\beta}(f) \in P(p,\alpha,\gamma) \\ &\Leftrightarrow (p+1)^{-\gamma}D^{\gamma}(J_{p,1}^{\gamma-\beta}(f)) \in P(p,\alpha) \\ &\Leftrightarrow (p+1)^{-\beta}D^{\beta}f \in P(p,\alpha) \\ &\Leftrightarrow f \in P(p,\alpha,\beta), \end{aligned}$$

and the proof of Corollary 2.2 is completed.

Next we define a function $h \in A_p$ by

$$h(z) = z^p + \sum_{n=1}^{\infty}\left(\frac{n+p+1}{p+1}\right)z^{n+p} \qquad (z \in U). \tag{2.18}$$

Then, in terms of the Hadamard product defined by (1.4), we have

$$(h*f)(z) = \frac{1}{p+1}\{f(z) + zf'(z)\} \tag{2.19}$$

which, when compared with (1.11) with $m = 1$, yields

$$(h*f)(z) = \frac{1}{p+1}D^1 f. \tag{2.20}$$

We now need the following lemma for another inclusion property of the class $P(p,\alpha,\beta)$.

LEMMA 2.2([8]). Let $\phi(u,v)$ be a complex valued function such that

$$\phi: D \to C, \qquad D \subset C \times C (C \textit{ is the complex plane}),$$

and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\phi(u,v)$ satisfies

(i) $\phi(u,v)$ is continuous in D,

(ii) $(1,0) \in D$ and $Re\{\phi(1,0)\} > 0$,

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \le -\frac{1+u_2^2}{2}$, $\qquad Re\{\phi(iu_2, v_1)\} \le 0$.

Let $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ be analytic in the unit disk U such that $(p(z), zp'(z)) \in D$ for all $z \in U$. If

$$Re\{(p(z), zp'(z))\} > 0 \qquad (z \in U),$$

then $Re\{p(z)\} > 0 (z \in U)$.

THEOREM 2.3. If $0 \le \alpha < p$ and $\beta \ge 0$, then

$$P(p,\alpha,\beta+1) \subset P(p,\mu,\beta) \qquad \left(\mu = \frac{2\alpha(p+1)+p}{2(p+1)+1}\right). \tag{2.21}$$

PROOF. Let the function

$$F(z) = \frac{1}{p+1}\{f(z) + zf'(z)\} \qquad (f \in A_p). \tag{2.22}$$

First, we shall show that

$$Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > \frac{2\alpha(p+1)+p}{2(p+1)+1} \qquad (0 \le \alpha < p, z \in U), \tag{2.23}$$

whenever

$$Re\left\{\frac{F'(z)}{z^{p-1}}\right\} > \alpha \qquad (0 \le \alpha < p, z \in U). \tag{2.24}$$

By the differentiation of $F(z)$, we obtain

$$F'(z) = \frac{1}{p+1}\{2f'(z) + zf''(z)\}. \tag{2.25}$$

We define the function $p(z)$ by

$$\frac{f'(z)}{pz^{p-1}} = \gamma + (1-\gamma)p(z) \tag{2.26}$$

with $\gamma = \frac{2\alpha(p+1)+p}{2p(p+1)+p}$ $(0 \le \gamma < 1)$. Then $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ is analytic in U. By using (2.25) and (2.26), we obtain

$$\frac{F'(z)}{z^{p-1}} = \frac{1}{p+1}\{(p^2+p)(\gamma + (1-\gamma)p(z)) + p(1-\gamma)zp'(z)\}. \tag{2.27}$$

Hence, in view of $Re\left\{\frac{F'(z)}{z^{p-1}}\right\} > \alpha$ $(0 \le \alpha < p)$, we have

$$Re\{\phi(p(z), zp'(z))\} > 0, \tag{2.28}$$

where $\phi(u,v)$ is defined by

$$\phi(u,v) = \frac{1}{p+1}\{(p^2+p)(\gamma + (1-\gamma)u) + p(1-\gamma)v\} - \alpha \tag{2.29}$$

with $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Then we see that

(i) $\phi(u,v)$ is continuous in $D = C \times C$,

(ii) $(1,0) \in D$ and $Re\{\phi(1,0)\} = p - \alpha > 0$,

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \le \frac{-(1+u_2^2)}{2}$,

$$Re\{\phi(iu_2, v_1)\} = \frac{1}{p+1}\{(p^2+p)\gamma + p(1-\gamma)v_1\} - \alpha$$

$$\le \frac{1}{p+1}\left\{(p^2+p)\gamma - \frac{p(1-\gamma)(1+u_2^2)}{2}\right\} - \alpha \le 0$$

for $\gamma = \frac{2\alpha(p+1)+p}{2p(p+1)+p}$. Consequently, $\phi(u,v)$ satisfies the conditions in Lemma 2.2. Therefore, we have

$$Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > p\gamma = \frac{2\alpha(p+1)+p}{2(p+1)+1}. \tag{2.30}$$

Next, in view of (2.20) and above arguments, we have

$$\begin{aligned} f \in P(p,\alpha,\beta+1) &\Leftrightarrow (p+1)^{-\beta-1}D^{\beta+1}f \in P(p,\alpha) \\ &\Rightarrow h * \{(p+1)^{-\beta}D^{\beta}f\} \in P(p,\alpha) \\ &\Rightarrow (p+1)^{-\beta}D^{\beta}f \in P(p,\mu) \qquad \left(\mu = \frac{2\alpha(p+1)+p}{2(p+1)+1}\right) \\ &\Leftrightarrow f \in P(p,\mu,\beta), \end{aligned} \tag{2.31}$$

which evidently proves Theorem 2.3.

REMARK. Since $0 \leq \; = \alpha < p$, we have

$$\mu = \frac{2\alpha(p+1)+p}{2(p+1)+1} > \alpha,$$

and hence $P(p,\mu,\beta) \subset P(p,\alpha,\beta)$.

3. THE CONVERSE PROBLEM.

Let T_p denote the class of functions of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad (p \in N = \{1,2,3,\ldots\}, a_{n+p} \geq 0)$$

which are analytic in U and let $T_p(\alpha,\beta) = T_p \cap P(p,\alpha,\beta)$.

In this section, we investigate the converse problem of integrals defined by (1.5) for the class $T_p(\alpha,\beta)$.

LEMMA 3.1. Let $f \in T_p$. Then $f \in T_p(\alpha,\beta)$ if and only if

$$\sum_{n=1}^{\infty} (n+p)\left(\frac{n+p+1}{p+1}\right)^{\beta} a_{n+p} \leq p - \alpha. \tag{3.1}$$

PROOF. Suppose that

$$\sum_{n=1}^{\infty} (n+p)\left(\frac{n+p+1}{p+1}\right)^{\beta} a_{n+p} \leq p - \alpha.$$

It is sufficient to show that the values for $\frac{(p+1)^{-\beta}(D^{\beta}f)'}{z^{p-1}}$ lie in a circle centered at p whose radius is $p-\alpha$. Indeed, we have

$$\left|\frac{(p+1)^{-\beta}(D^{\beta}f)'}{z^{p-1}} - p\right| = \left| - \sum_{n=1}^{\infty} (n+p)\left(\frac{n+p+1}{p+1}\right)^{\beta} a_{n+p}\, z^n\right|$$

$$\leq \sum_{n=1}^{\infty} (n+p)\left(\frac{n+p+1}{p+1}\right)^{\beta} a_{n+p} |z|^n$$

$$< \sum_{n=1}^{\infty} (n+p)\left(\frac{n+p+1}{p+1}\right)^{\beta} a_{n+p} \leq p - \alpha. \tag{3.2}$$

Conversely, assume that

$$Re\left\{\frac{(p+1)^{-\beta}(D^{\beta}f)'}{z^{p-1}}\right\} > \alpha (0 \leq \alpha < p), \tag{3.3}$$

which is equivalent to

$$Re\left\{\sum_{n=1}^{\infty} (n+p)\left(\frac{n+p+1}{p+1}\right)^{\beta} a_{n+p}\, z^n\right\} < p - \alpha. \tag{3.4}$$

Choose values of z on the real axis so that

$$\sum_{n=1}^{\infty} (n+p)\left(\frac{n+p+1}{p+1}\right)^{\beta} a_{n+p}\, z^n$$

is real. Letting $z \to 1$ along the real axis, we obtain

$$\sum_{n=1}^{\infty} (n+p)\left(\frac{n+p+1}{p+1}\right)^{\beta} a_{n+p} \leq p - \alpha.$$

The proof is completed.

THEOREM 3.1. Let $F \in T_p(\alpha,\beta)$ and $f(z) = \left[\frac{z^{1-c}}{p+c}\right][z^cF(z)]'$ $(c \in N)$. Then the function $f(z)$ belongs to the class $T_p(\delta,\beta)$ $(0 \leq \delta < p)$ for $|z| < r$, where

$$r = \inf_{n \geq 1} \left[\frac{(p-\delta)(p+c)}{(p-\alpha)(n+p+c)}\right]^{\frac{1}{n}}. \tag{3.5}$$

The result is sharp.

PROOF. Let $F(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p}$. Then it follows from (1.5) that

$$\begin{aligned} f(z) &= \left[\frac{z^{1-c}}{p+c}\right]\frac{d}{dz}[z^cF(z)] \\ &= z^p - \sum_{n=1}^{\infty}\left(\frac{n+p+c}{p+c}\right)a_{n+p} z^{n+p}. \end{aligned} \tag{3.6}$$

To prove the result, it suffices to show that

$$\left|\frac{(p+1)^{-\beta}(D^\beta f)'}{z^{p-1}} - p\right| \leq p - \delta \tag{3.7}$$

for $|z| \leq r$. Now

$$\begin{aligned} \left|\frac{(p+1)^{-\beta}(D^\beta f)'}{z^{p-1}} - p\right| &= \left| - \sum_{n=1}^{\infty}(n+p)\left(\frac{n+p+1}{p+1}\right)^\beta\left(\frac{n+p+c}{p+c}\right)a_{n+p} z^n\right| \\ &\leq \sum_{n=1}^{\infty}(n+p)\left(\frac{n+p+1}{p+1}\right)^\beta\left(\frac{n+p+c}{p+c}\right)a_{n+p}|z|^n. \end{aligned} \tag{3.8}$$

Thus we have

$$\left|\frac{(p+1)^{-\beta}(D^\beta f)'}{z^{p-1}} - p\right| \leq p - \delta \tag{3.9}$$

if

$$\sum_{n=1}^{\infty}(n+p)\left(\frac{n+p+1}{p+1}\right)^\beta\left(\frac{n+p+c}{p+c}\right)a_{n+p}|z|^n \leq p-\delta. \tag{3.10}$$

But Lemma 3.1 confirms that

$$\sum_{n=1}^{\infty}(n+p)\left(\frac{n+p+1}{p+1}\right)^\beta a_{n+p} \leq p-\alpha. \tag{3.11}$$

Therefore (3.10) will be satisfied if

$$\left(\frac{n+p}{p-\delta}\right)\left(\frac{n+p+c}{p+c}\right)|z|^n \leq \left(\frac{n+p}{p-\alpha}\right) \tag{3.12}$$

for each $n \in N$, or if

$$|z| \leq \left[\left(\frac{p-\delta}{p-\alpha}\right)\left(\frac{p+c}{n+p+c}\right)\right]^{\frac{1}{n}}. \tag{3.13}$$

The required result follows now from (3.13). Sharpness follows if we take

$$F(z) = z^p - \left(\frac{p-\alpha}{n+p}\right)\left(\frac{p+1}{n+p+1}\right)^\beta z^{n+p} \tag{3.14}$$

for each $n \in N$.

THEOREM 3.2. Let $F \in T_p(\alpha,\beta)$ and $f(z) = \left[\frac{z^{1-c}}{p+c}\right][z^cF(z)]'$ $(c \in N)$. Then the function $f(z)$ p-valently convex of order δ $(0 \leq \delta < p)$ in the disk

$$|z| < r^* = \inf_{n \geq 1}\left[\frac{p(p-\delta)}{(n+p+\delta)(p-\alpha)}\left(\frac{n+p+c}{p+c}\right)\left(\frac{n+p+1}{p+1}\right)^{\beta}\right]^{\frac{1}{n}}. \tag{3.15}$$

The result is sharp.

PROOF. To prove the theorem, it is sufficient to show that

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right| \leq p-\delta \tag{3.16}$$

for $|z| \leq r^*$. In view of (3.6), we have

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right| = \left|\frac{-\sum_{n=1}^{\infty} n(n+p)\left(\frac{n+p+c}{p+c}\right)a_{n+p}\, z^{n+p-1}}{(p-\sum_{n=1}^{\infty}(n+p)\left(\frac{n+p+c}{p+c}\right)a_{n+p}\, z^{n})z^{p-1}}\right|$$

$$\leq \frac{\sum_{n=1}^{\infty} n(n+p)\left(\frac{n+p+c}{p+c}\right)a_{n+p}|z|^{n}}{p-\sum_{n=1}^{\infty}(n+p)\left(\frac{n+p+c}{p+c}\right)a_{n+p}|z|^{n}}. \tag{3.17}$$

Thus

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right| \leq p-\delta \tag{3.18}$$

if

$$\frac{\sum_{n=1}^{\infty} n(n+p)\left(\frac{n+p+c}{p+c}\right)a_{n+p}|z|^{n}}{p-\sum_{n=1}^{\infty}(n+p)\left(\frac{n+p+c}{p+c}\right)a_{n+p}|z|^{n}} \leq p-\delta, \tag{3.19}$$

or

$$\sum_{n=1}^{\infty}\frac{(n+p)(n+p+\delta)}{p(p-\delta)}\left(\frac{n+p+c}{p+c}\right)a_{n+p}|z|^{n} \leq 1. \tag{3.20}$$

But from Lemma 3.1, we obtain

$$\sum_{n=1}^{\infty}\left(\frac{n+p}{p-\alpha}\right)\left(\frac{n+p+1}{p+1}\right)^{\beta} a_{n+p} \leq 1. \tag{3.21}$$

Hence $f(z)$ is p-valently convex of order δ $(0 \leq \delta < p)$ if

$$\frac{(n+p)(n+p+\delta)}{p(p-\delta)}\left(\frac{n+p+c}{p+c}\right)|z|^{n} \leq \left(\frac{n+p}{p-\alpha}\right)\left(\frac{n+p+1}{p+1}\right)^{\beta}, \tag{3.22}$$

or

$$|z| \leq \left[\frac{p(p-\delta)}{(n+p+\delta)(p-\alpha)}\left(\frac{p+c}{n+p+c}\right)\left(\frac{n+p+1}{p+1}\right)^{\beta}\right]^{\frac{1}{n}} \tag{3.23}$$

for each $n \in N$. This completes the proof of the theorem. The result is sharp for the function given by (3.14).

REFERENCES

1. AOUF, M.K., OBRADOVIC, M. & OWA, S., On certain class of p-valent functions with reference to the Bernardi integral operator, Math. Japonica 35 (1990), 839-848.
2. BERNARD, S.D., Convex and starlike univalent functions, Trans. Amer. Math. Soc. 135 (1969), 429-446.

3. FLETT, T.M., The dual of an inequality of Hardy and Littlewood and some related inequalities, J. Math. Anal. Appl. 38 (1972), 746-765.

4. KIM, Y.C., LEE, S.H., & SRIVASTAVA, H.M., Some properties of convolution operators in the class $P_\alpha(\beta)$, submitted.

5. LIBERA, R.J., Some classes of regular univalent functions, Proc. Amer. Math. Soc. 16 (1965), 755-758.

6. LIVINGSTON, A.E., On the radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. 17 (1966), 352-357.

7. MACGREGOR, T.H., Functions whose derivatives has a positive real part, Trans. Amer. Math. Soc. 104 (1962), 532-537.

8. MILLER, S.S., Differential inequalities and Caratheodory functions, Bull. Amer. Math. Soc. 81 (1975), 79-81.

9. OWA, S. & NUNOKAWA, M., Properties of certain analytic functions, Math. Japonica 33 (1988), 577-582.

10. REDDY, G.L. & PADMANABHAN, K.S., On analytic functions with reference to the Bernardi integral operator, Bull. Austral. Math. Soc. 25 (1982), 387-396.

11. UMEZAWA, T., Multivalently close-to-convex functions, Proc. Amer. Math. Soc. 8 (1957), 869-874.