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GENERIC SUBMANIFOLDS OF A LOCALLY CONFORMAL KAEHLER MANIFOLD-II

M. HASAN SHAHID

Department of Mathematics
Faculty of Natural Sciences
Jamia Millia Islamia
New Delhi-110 025, India

and

KOUEI SEKIGAWA

Department of Mathematics
Faculty of Science
Niigata University
Niigata, 950-21, Japan

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ABSTRACT. The purpose of this paper is to study generic submanifolds with parallel structures, generic product submanifolds and totally umbilical submanifolds of a locally conformal Kaehler manifold. Moreover, we give some examples of generic submanifolds of a locally conformal Kaehler manifold which are not CR-submanifolds.

KEY WORDS AND PHRASES. Locally conformal Kaehler manifold, generic submanifold, CR-submanifold.
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1. INTRODUCTION.

Let $\bar{M}$ be an almost Hermitian manifold with almost Hermitian structure (J,g). The manifold $\bar{M}$ is called a *local conformal Kaehler* (briefly, l.c.K.) manifold if for any $x \in \bar{M}$ there is an open neighborhood $\mathcal{U}$ such that, for some differentiable function $\sigma: \mathcal{U} \to \mathbf{R}, g' = e^{-\sigma} g|_{\mathcal{U}}$ is a Kaehler metric on $\mathcal{U}$. If $\mathcal{U} = \bar{M}$ then the manifold is called a *globally conformal Kaehler* (briefly, g.c.K.) manifold. Let Ω be the Kaehler form of an almost Hermitian manifold $\bar{M}$ defined by $\Omega(U,V) = g(U,JV)$, for any vector fields U,V on $\bar{M}$. Then it is easy to see that $\bar{M}$ is a l.c.K. manifold if and only if there is a global 1-form ω (the Lee form of $\bar{M}$) such that

$$d\Omega = \omega \wedge \Omega, \qquad d\omega = 0, \tag{1.1}$$

and $\bar{M}$ is a g.c.K. manifold if and only if ω is exact. For a l.c.K. manifold $\bar{M}$, the Lee vector field B is given by

$$g(B,U) = \omega(U) \tag{1.2}$$

for any vector field U on $\bar{M}$. We denote by $\bar{\nabla}$ the Levi-Civita connection of g. We define a torsion-free linear connection $\tilde{\bar{\nabla}}$ on $\bar{M}$ by

$$\tilde{\bar{\nabla}}_U V = \bar{\nabla}_U V - \frac{1}{2}\{\omega(U)V + \omega(V)U - g(U,V)B\} \tag{1.3}$$

for any vector fields U,V on $\bar{M}$. The linear connection $\tilde{\nabla}$ is called the Weyl connection of $\bar{M}$. Then we may easily observe that the Weyl connection $\tilde{\nabla}$ satisfies the condition: $\tilde{\nabla} J = 0$, $\tilde{\nabla} g = 0$ on each neighborhood on which $(J, g' = e^{-\sigma} g|_{\mathcal{U}})$ is a Kaehler structure.

In general, let $\bar{M}$ be a 2n-dimensional almost Hermitian manifold and M be an m-dimensional Riemman manifold isometrically immersed in $\bar{M}$. Let ∇ be the Levi-Civita connection on M induced by $\bar{\nabla}$. Then the Gauss and Weingarten formulas are given respectively by

$$\bar{\nabla}_U V = \nabla_U V + h(U,V), \tag{1.4}$$

$$\bar{\nabla}_U N = -A_N U + \nabla^{\perp}_U N \tag{1.5}$$

for any vector fields U,V tangent to M and N normal to M, where h is the second fundamental form of M in $\bar{M}$ and $\nabla^{\perp}$ is the normal connection on the normal bundle $T^{\perp}(M)$ with respect to the Levi-Civita connection $\bar{\nabla}$. Then we have $g(A_N U,V) = g(h(U,V),N)$, for any vector fields U,V tangent to M. For any vector field U tangent to M, we put

$$JU = PU + FU \tag{1.6}$$

where PU and FU are tangential and normal components of JU, respectively. Then P is an endomorphism of the tangent bundle $T(M)$ of M and F is a normal bundle valued 1-form on $T(M)$. For any vector field N normal to M, we put

$$JN = tN + fN, \tag{1.7}$$

where tN and fN are the tangential and normal components of JN, respectively. Then f is an endomorphism of the normal bundle $T^{\perp}(M)$ of M in $\bar{M}$ and t is a tangent bundle valued 1-form on $T^{\perp}(M)$.

DEFINITION: Let M be a submanifold of an almost Hermitian manifold $\bar{M}$. The holomorphic subspace D_x of T_xM at $x \in M$ is defined by $D_x = T_xM \cap JT_xM$. D_x is the maximal complex subspace of $T_x\bar{M}$ which is contained in T_xM. If the dimension of D is constant along M, and furthermore, D defines a differentiable distribution on M, then M is called a generic submanifold of $\bar{M}$.

Let M be a generic submanifold of an almost Hermitian manifold $\bar{M}$. We call the distribution D the holomorphic distribution and the orthogonal complementary distribution $D^{\perp}$ the purely real distribution. They satisfy the following relations:

$$D_x \cap D_x^{\perp} = \{0\}, \qquad D_x^{\perp} \cap JD_x^{\perp} = \{0\} \text{ for each } x \in M.$$

Let ν_x be the holomorphic normal space of M at x, i.e.,

$$\nu_x = T_x^{\perp} M \cap JT_x^{\perp} M.$$

Then $\nu_x (x \in M)$ defines a differentiable vector subbundle ν of $T^{\perp}(M)$ satisfying

$$T^{\perp}(M) = FD^{\perp} + \nu \text{ (direct sum)}, \; t(T^{\perp}(M)) = D^{\perp}. \tag{1.8}$$

Furthermore, we have

$$D \perp D^{\perp}, PD = D \text{ and } D^{\perp} \supset PD^{\perp}. \tag{1.9}$$

We put $dim\ D = 2p$ and $dim\ D^{\perp} = q$. If $p,\ q \geq 1$, then the generic submanifold M is said to be proper. In the sequel, we shall consider only proper generic submanifolds. We put

$$(\nabla_U P)V = \nabla_U(PU) - P(\nabla_U V), \tag{1.10}$$

and

$$(\nabla_U F)V = \nabla_U^{\perp}(FV) - F\nabla_U V \tag{1.11}$$

for any vector fields U,V tangent to M. We say that P (resp. F) is parallel if $(\nabla_U P)V = 0$ (resp. $(\nabla_U F)V = 0$) for any vector fields U,V tangent to M. If a generic submanifold M of an almost Hermitian manifold $\bar{M}$ satisfies the condition $JD^{\perp} \subset T^{\perp}(M)$, then M is called a CR-submanifold of $\bar{M}$. Dragomir ([4]) studied CR-submanifolds of l.c.K. manifolds. The present paper is a continuation of the previous work [5].

2. PRELIMINARIES.

Let M be a generic submanifold of a l.c.K. manifold $\bar{M}$. For the Lee vector field B of $\bar{M}$, we put

$$B = B^T + B^{\perp} \text{ along } M, \tag{2.1}$$

where B^T(resp. $B^{\perp}$) is the tangential (resp. normal) component of B. Furthermore, we put

$$B^T = B^D + B^{D^{\perp}} \text{ along } M, \tag{2.2}$$

where B^D (resp. $B^{D^{\perp}}$) is the D-component (resp. $D^{\perp}$-component) of $B^{\perp}$. Since $\tilde{\nabla} J = 0$ with respect to the Weyl connection $\tilde{\nabla}$, taking account of $(1.3) \sim (1.7)$, (1.11), (1.12), (2.1) and (2.2), we have

$$\begin{aligned}(\nabla_X P)Y &- \tfrac{1}{2}\omega(JY)X + \tfrac{1}{2}\omega(Y)JX - th(X,Y)\\ &+ \tfrac{1}{2}g(X,JY)B^T - \tfrac{1}{2}g(X,Y)PB^{\perp} - \tfrac{1}{2}g(X,Y)tB^{\perp} = 0,\end{aligned} \tag{2.3}$$

$$\begin{aligned}h(X,JY) &- F\nabla_X Y + \tfrac{1}{2}g(X,JY)B^{\perp}\\ &- \tfrac{1}{2}g(X,Y)FB^T - \tfrac{1}{2}g(X,Y)fB^{\perp} - fh(X,Y) = 0,\end{aligned} \tag{2.4}$$

$$(\nabla_X P)Z - A_{FZ}X - \tfrac{1}{2}\omega(JZ)X + \tfrac{1}{2}\omega(Z)JX - th(X,Z) = 0, \tag{2.5}$$

$$(\nabla_X F)Z + h(X,PZ) - fh(X,Z) = 0, \tag{2.6}$$

$$(\nabla_Z P)X - \tfrac{1}{2}\omega(JX)Z - \tfrac{1}{2}\omega(X)PZ - th(X,Z) = 0, \tag{2.7}$$

$$F\nabla_Z X - h(JX,Z) + fh(X,Z) = 0, \tag{2.8}$$

$$\begin{aligned}(\nabla_Z P)W &- A_{FW}Z - \tfrac{1}{2}\omega(JW)Z + \tfrac{1}{2}\omega(W)PZ + \tfrac{1}{2}g(Z,JW)B^T\\ &- \tfrac{1}{2}g(Z,W)PB^T - \tfrac{1}{2}g(Z,W)tB^{\perp} - th(Z,W) = 0,\end{aligned} \tag{2.9}$$

$$\begin{aligned}(\nabla_Z F)W &+ h(Z,PW) + \tfrac{1}{2}g(Z,JW)B^T + \tfrac{1}{2}\omega(W)FZ\\ &- \tfrac{1}{2}g(Z,W)FB^T - \tfrac{1}{2}g(Z,W)fB^{\perp} - fh(Z,W) = 0,\end{aligned} \tag{2.10}$$

for any $X,Y \in D$ and $Z,W \in D^{\perp}$.

We recall the conditions for the distributions D and $D^{\perp}$ to be integrable.

PROPOSITION 2.1 ([5]). The distribution $D^{\perp}$ is integrable if and only if

$$g(h(X,JY)-h(JX,Y)+g(X,JY)B,FZ)=0,$$

for any $X,Y\in D$ and $Z\in D^{\perp}$.

PROPOSITION 2.2 ([5]). The distribution $D^{\perp}$ is integrable if and only if

$$\nabla_Z(PW)-\nabla_W(PZ)+A_{FZ}W-A_{FW}Z+g(Z,JW)B\in D^{\perp},$$

for any $Z,W\in D^{\perp}$.

Let M be a totally geodesic generic submanifold of a Kaehler manifold $\bar{M}$. Then it follows immediately that P and F are parallel, and furthermore D is integrable. So, it is worthwhile to study generic submanifolds with parallel structures and also totally umbilical generic submanifolds in a l.c.K. manifold.

3. GENERIC SUBMANIFOLDS WITH PARELLEL STRUCTURES.

In this section, we consider generic submanifolds with parallel P (resp. F) of a l.c.K. manifold.

THEOREM 3.1. Let M be a generic submanifold of a l.c.K. manifold $\bar{M}$. If P is parallel, then D is integrable and $B^{D^{\perp}}=0$ along M. Moreover, if $dim\ D\geq 4$, then $B^T=0$ along M.

PROOF. By (1.11) and (2.3), we get

$$-\tfrac{1}{2}\omega(JY)X+\tfrac{1}{2}\omega(JX)Y+g(X,JY)B^T+\tfrac{1}{2}\omega(Y)JX-\tfrac{1}{2}\omega(X)JY=0, \quad (3.1)$$

for $X,Y\in D$. Putting $Y=JX$ in (3.1), we get

$$\omega(X)X+\omega(JX)JX-g(X,X)B^{\perp}=0, \quad (3.2)$$

for any vector field X on M. From (3.2), we get

$$(p-1)g(B^D,B^D)+pg(B^{D^{\perp}},B^{D^{\perp}})=0. \quad (3.3)$$

First, we assume $p\geq 2$. Then, by (3.3), we have

$$B^D=0,\ B^{D^{\perp}}=0 \text{ (and hence } B^{\perp}=0). \quad (3.4)$$

Thus, by (2.3) and (3.4), we get

$$2\ th(X,Y)+g(X,Y)tB=0, \quad (3.5)$$

for $X,Y\in D$. On one hand, by (1.11) and (2.4), we get

$$F\nabla_X(PY)+h(X,Y)+fh(X,JY)+\tfrac{1}{2}g(X,Y)B^{\perp}+\tfrac{1}{2}g(X,JY)fB^{\perp}=0, \quad (3.6)$$

for $X,Y\in D$. By (1.11) and (3.6), we get

$$FP[X,Y]+f\{h(X,JY)-h(JX,Y)\}+g(X,JY)fB^{\perp}=0, \quad (3.7)$$

for $X,Y\in D$. From (3.5), we get also

$$t\{h(X,JY)-h(JX,Y)\}+g(X,JY)tB^{\perp}=0, \quad (3.9)$$

for $X,Y\in D$. Thus, by (3.7) and (3.8), we have

$$J\{h(X,JY)-h(JX,Y)\}+g(X,JY)JB^{\perp}=-FP[X,Y], \quad (3.9)$$

for $X,Y\in D$. By (3.9), we have

$$g(h(X,JY)-h(JX,Y)+g(X,JY)B,JZ)$$
$$=g(FP[X,Y],Z)=0, \tag{3.10}$$

for $X,Y\in D$ and $Z\in D^{\perp}$. Thus, from Proposition 3.1 and (3.10), it follows that D is integrable. Next, we assume that $p=1$. Then, by (3.3), we have

$$B^{D^{\perp}}=0. \tag{3.11}$$

By (2.3), we get

$$\frac{1}{2}\omega(Y)X-\frac{1}{2}\omega(X)Y+\frac{1}{2}\omega(JY)JX-\frac{1}{2}\omega(JX)JY$$
$$-t\{h(X,JY)-h(JX,Y)\}-g(X,JY)PB^{T}-g(X,JY)tB^{T}=0, \tag{3.12}$$

for $X,Y\in D$. On one hand, by (2.4) and (3.1), we get

$$FP[X,Y]-f\{h(X,JY)-h(JX,Y)\}+g(X,JY)fB^{\perp}=0, \tag{3.13}$$

for $X,Y\in D$. By (3.12) and (3.13), we get

$$J\{h(X,JY)-h(JX,Y)\}+g(X,JY)JB^{T}+g(X,JY)PB^{T}$$
$$+\frac{1}{2}\omega(X)Y-\frac{1}{2}\omega(Y)X+\frac{1}{2}\omega(JX)JY-\frac{1}{2}\omega(JY)JX+FP[X,Y]=0, \tag{3.14}$$

for $X,Y\in D$. From (3.11), it follows that $PB^{T}=JB^{T}$.
Thus, (3.14) implies

$$h(X,JY)-h(JX,Y)+g(X,JY)B$$
$$=\frac{1}{2}\omega(X)JY-\frac{1}{2}\omega(Y)JX-\frac{1}{2}\omega(JX)Y+\frac{1}{2}\omega(JY)X+JFP\{X,Y], \tag{3.15}$$

for $X,Y\in D$. By (3.15), we have

$$g(h(X,JY)-h(JX,Y)+g(X,JY)B,FZ)=g(FP[X,Y],Z)=0, \tag{3.16}$$

for $X,Y\in D$ and $Z\in D^{\perp}$. Thus, from (3.16) and Proposition 3.1, if follows that D is integrable.

THEOREM 3.2. Let M be a generic submanifold of a l.c.K. manifold $\bar{M}$ such that F is parallel. Then the distribution D is integrable and each leaf of D is totally geodesic in M.

PROOF. By (1.12), we have

$$0=(\nabla_{K}F)Y=F\nabla_{X}Y,\ \text{for}\ X,Y\in D. \tag{3.17}$$

By (3.17), we have $\nabla_{X}Y\in D$ for any $X,Y\in D$, from which the theorem follows immediately.

4. GENERIC PRODUCT SUBMANIFOLDS.

Let M be a generic submanifold of an almost Hermitian manifold $\bar{M}$. If M is locally expressed in the form $M=M_{D}\times M_{D^{\perp}}$, where M_{D} (resp. $M_{D^{\perp}}$) is a holomorphic submanifold (resp. a purely real submanifold) of $\bar{M}$, then M is called a generic product submanifold of $\bar{M}$. In this section, we consider generic product submanifold of a l.c.K. manifold $\bar{M}$.

THEOREM 4.1. Let M be a generic product submanifold of a l.c.K. manifold $\bar{M}$. If $B^{D}=0$ along M, then we have

$$B^{T}=0\ \text{along}\ M, \tag{4.1}$$

and

$$\nabla_{X}P=0, \qquad (\nabla_{Z}P)X=0, \tag{4.2}$$

for $X\in D, Z\in D^{\perp}$.

PROOF. Since $(\nabla_{X}P)Z\in D^{\perp}$, for $X\in D, Z\in D^{\perp}$, by (2.5), we get

$$g(h(X,Y),FZ)+\frac{1}{2}\omega(JZ)g(X,Y)-\frac{1}{2}\omega(Z)g(JX,Y)=0, \tag{4.3}$$

for $X, Y \in D$, $Z \in D^{\perp}$. By (4.3), we get immediately $B^{D^{\perp}} = 0$, and hence (4.1). Since $(\nabla_X P)Y \in D$, for $X, Y \in D$, by (2.3) and (4.1), we get

$$(\nabla_X P)Y = 0, \qquad \text{for } X, Y \in D. \tag{4.4}$$

Since $(\nabla_Z P) \in D^{\perp}$, for $Z, W \in D^{\perp}$, by (2.9) and (4.1), we get

$$g(h(X,Z), FW) = 0, \text{ for } X \in D,\ Z \in D^{\perp}. \tag{4.5}$$

by (2.5), (4.1) and (4.5), we have

$$\begin{aligned} 0 &= g((\nabla_X P)Z, W) - g(h(X,W), FZ) - g(th(X,Z), W) \\ &= g((\nabla_X P)Z, W) - g(h(X,W), FZ) + g(h(X,Z), FW) \\ &= g((\nabla_X P)Z, W) \end{aligned} \tag{4.6}$$

for $X \in D$, Z, $W \in D^{\perp}$. By (4.3) and (4.6), we have the first equality of (4.2). Since $(\nabla_Z P)X \in D$, for $X \in D$, $Z \in D^{\perp}$, by (2.7), we have immediately the second equality of (4.2). *Q.E.D.*

COROLLARY 4.2. Let M be a CR-product submanifold of a l.c.K. manifold $\bar{M}$. If $B^D = 0$ along M, then P is parallel.

PROOF. Since $PW = 0$, and $\nabla_Z W$, $(\nabla_Z P)W \in D^{\perp}$, for Z, $W \in D^{\perp}$, we have immediately $(\nabla_Z P)W = 0$ for Z, $W \in D^{\perp}$. Thus, from this together with (4.2), the corollary follows. *Q.E.D.*

5. TOTALLY UMBILICAL GENERIC SUBMANIFOLDS.

A Riemmannian submanifold M of a Riemannian manifold $\bar{M}$ is called a totally umbilical submanfiold if

$$h(U,V) = g(U,V)H, \tag{5.1}$$

for any vector fields U, V tangent to M, where H is the mean curvature vector. In this section, we consider some totally umbilical generic submanifolds of a l.c.K. manifold.

THEOREM 5.1. Let M be a totally umbilical generic submanifold of a l.c.K. manifold $\bar{M}$ such that P is parallel. Then we have $B^{D^{\perp}} = 0$ and $2H + B^{\perp} = 0$ along M. In particular, if $\dim D \geq 4$, then $2H + B = 0$ along M.

PROOF. Since P is parallel, from Theorem 3.1 and (3.4), (3.11), it follows that D is integrable and

$$B^{D^{\perp}} = 0. \tag{5.2}$$

By (2.4), we have easily

$$2H + B^{\perp} = 0. \tag{5.3}$$

By (3.1), we get

$$\omega(X)^2 + \omega(JX)^2 = g(X,X)g(B^T, B^T), \qquad \text{for } X \in D. \tag{5.4}$$

By (5.2) and (5.4), we have

$$(p-1)g(B^T, B^T) = 0. \tag{5.5}$$

By (5.5), if $p \geq 2$, we have $B^T = 0$. Therefore, the Theorem follows from (5.3). *Q.E.D.*

COROLLARY 5.2. Let M be a totally umbilical generic submanifold of a l.c.K. manifold $\bar{M}$ such that $B \in D$. If P is parallel, then M is totally geodesic and $B = 0$ along M.

THEOREM 5.3. Let M be a totally umbilical generic submanifold of a l.c.K. manifold $\bar{M}$ such that $dim\ FD^{\perp} < dim\ D^{\perp}$ on a dense open subset in M. If P is parallel and $dim\ D^{\perp} \geq 2$, then $2H + B = 0$ along M.

PROOF. By (1.5), (2.9), (5.1) and (5.2), we have

$$\begin{aligned} 0 &= -\tfrac{1}{2}\omega(JW)g(Z,B^T) + \tfrac{1}{2}\omega(JZ)g(W,B^T) \\ &\quad + \tfrac{1}{2}\omega(W)g(PZ,B^T) - \tfrac{1}{2}\omega(Z)g(PW,B^T) + g(Z,JW)g(B^T,B^T) \\ &= g(Z,JW)g(B^T,B^T), \end{aligned} \tag{5.6}$$

for $Z, W \in D^{\perp}$. From (5.2) and (5.6), taking account of Theorem 5.1, the theorem follows immediately. *Q.E.D.*

THEOREM 5.4. Let M be a totally umbilical generic submanifold of a l.c.K. manifold $\bar{M}$ such that $B \in D^{\perp}$. Then the purely real distribution $D^{\perp}$ is totally geodesic in M.

PROOF. For $X \in D$, $W \in D^{\perp}$ and $N \in T^{\perp}(M)$, by (1.3), (1.5) and (5.1), we have

$$\begin{aligned} 0 &= g((\tilde{\bar{\nabla}}_W J)N, X) \\ &= g(\tilde{\bar{\nabla}}_W(JN), X) - g(J\tilde{\bar{\nabla}}_W N, X) \\ &= g(\tilde{\bar{\nabla}}_W(JN), X) + g(\bar{\nabla}_W N, JX) \\ &= g(\bar{\nabla}_W(tN), X) + g(\bar{\nabla}_W(fN), X) \\ &= g(\nabla_W(tN), X), \end{aligned}$$

from which the theorem follows immediately. *Q.E.D.*

THEOREM 5.5. Let M be a totally umbilical generic submanifold of a l.c.K. manifold $\bar{M}$ such that F is parallel. Then we have $2H + B^{\perp} = 0$ along M.

PROOF. Since F is parallel, from Theorem 3.2, it follows that $D^{\perp}$ is integrable and each leaf of $D^{\perp}$ is totally geodesic in M. Thus, by (2.4) and (5.1), we have immediately $2H + B^{\perp} = 0$. *Q.E.D.*

6. EXAMPLES.

In this section, we give some examples of generic submanifolds of Hopf manifolds which are not CR-submanifolds. Let $\mathbf{R}^{2n+2}$ be a $(2n+2)$-dimensional Euclidean space equipped with the canonical inner product (,) and $\{e_1, ..., e_{2n+1}, e_{2n+2}\}$ the canonical orthonormal basis of $\mathbf{R}^{2n+2}$. We denote by J_0 the complex structure on $\mathbf{R}^{2n+2}$ defined by

$$J_0 e_{2m-1} = e_{2m}, J_0 e_{2m} = -e_{2m-1}, \qquad 1 \leq m \leq n+1. \tag{6.1}$$

Let $S^{2n+1} = \{x \in \mathbf{R}^{2n+2}; (x,x) = 1\}$ be a $(2n+1)$-dimensional unit sphere with the canonical Sasakian structure (φ, ξ, η, h) induced from the Kaehler structure $(J_0, (,))$ on $\mathbf{R}^{2n+2}$. It is well known that the structure vector field ξ defines the Hopf fibration $\pi: S^{2n+1} \to \mathbf{C}P^n$, where $\mathbf{C}P^n$ is a (complex) n-dimensional complex projective space equipped with the canonical Fibini-Study metric of constant holomorphic sectional curvature 4. Let $S^1 = \{e^{t\sqrt{-1}}; t \in \mathbf{R}\}$ be a unit circle. We define an almost complex structure J on $M = S^{2n+1} \times S^1$ (resp. $\bar{M} = S^{2n+1} \times \mathbf{R}$) by

$$JT = \xi,\ J\xi = -T \text{ and } \qquad JU = \varphi U, \tag{6.2}$$

for any vector field U on $\bar{M}$ such that $\eta(U) = 0$, where $T = \frac{\partial}{\partial t}$ is the canonical unit vector field on S^1 (resp. $\mathbf{R}^1$). Then $(S^{2n+1} \times S^1, J)$ (resp. $(S^{2n+1} \times \mathbf{R}^1, J)$) is a l.c.K. manifold (resp. a g.c.K. manifold) together with the product metric $g = h + 1$ on $\bar{M} = S^{2n+1} \times S^1$ (resp. $\bar{M} = S^{2n+1} \times \mathbf{R}^1$). Then the Lee form ω of $\bar{M}$ is given by $\omega = 2dt$.

I. We denote by S_{pq} the Segre imbedding $S_{pq}: \mathbf{C}P^p \times \mathbf{C}P^q \to \mathbf{C}P^{p+q+pq}$ ([2]). Let M_1 be any q-dimensional purely real submanifold of $\mathbf{C}P^q$. Then $M = \mathbf{C}P^p \times M_1$ is a generic product submanifold of CP^{p+q+pq} in which CP^p is imbedded as a totally geodesic complex submanifold. We denote by the immersion $\iota: M_1 \to CP^q$. Let $M = \{S_{pq} \circ (1 \times \iota)^{-1}(S^{2(p+q+pq)+1})$ be pull-back of the Hopf bundle $\pi: S^{2(p+q+pq)+1})$ by the immersion $S_{pq} \circ (1 \times \iota): CP^p \times M_1 \to CP^{p+q+pq}$. Then we may easily observe that M is a generic submanifold of the Hopf manifold $\bar{M} = S^{2(p+q+pq)+1} \times S^1$. For example, let M_1 be the real submanifold of CP^q $(q > 1)$ defined by $M_1 = \{(x_0, ..., x_{q-1}, x_q + \sqrt{-1}\ \ x_{q-1}) \in CP^q;\ (x_0, ..., x_{q-1}, x_q)$ are homogeneous coordinates of a q-dimensional real projective space $\mathbf{R}P^q\}$. Then M_1 is a purely real submanifold of CP^q which is not totally real.

In the following II $\sim$ IV, we assume that $\bar{M} = S^7 \times S^1$.

II. Let Π be the 5-dimensional linear subspace of $\mathbf{R}^8$ given by $\Pi = span_{\mathbf{R}}\{e_1, ..., e_5\}$. We put $S^4 = S^7 \cap \Pi$ and $M_2^4 = \{x = \sum_{i=1}^{5} x_i e_i \in S^4;\ 0 < x^5 < 1\}$. For each point $x \in M_2^4$, let D'_x be the subspace of $T_x M_2^4$ defined by $D'_x = \{u \in T_x M_2^4;\ (u, J_0 x) = 0,\ (u, e_5) = 0\}$. We put $M = M_2^4 \times S^1 (\subset S^7 \times S^1)$. For each point $(x, e^{\sqrt{-1}t}) \in M$, let $D_{(x, e^{\sqrt{-1}t})}$ be the subspace of $T_{(x, e^{\sqrt{-1}t})}M$ defined by $D_{(x, e^{\sqrt{-1}t})} = \{(u, 0) \in T_{(x, e^{\sqrt{-1}t})}M;\ u \in D'_x\}$. Then we may easily observe that M is a totally geodesic generic submanifold of $\bar{M}$ with the holomorphic distribution D which is not a CR-submanifold of $\bar{M}$. We may easily check that the Lee form of $\bar{M}$ is tangent to M.

III. We put $M = M_2^4 \times \{1\}$ $(\subset S^7 \times S^1)$. Then M is also a totally geodesic generic submanifold of $\bar{M}$ with holomorphic distribution D as in II (restricted to $M_2^4 \times \{1\}$) which is not CR-submanifold of $\bar{M}$. In this case, we may easily check that the Lee form of $\bar{M}$ is normal to M.

IV. We put $M_3^4 = \{x = \sum_{i=1}^{5} x_i e_i + \frac{1}{\sqrt{2}} e_7 \in S^7;\ 0 < x_5 < \frac{1}{\sqrt{2}}\}$. For each point $x \in M_3^4$, let D''_x be the subspace of $T_x M_3^4$ defined by $D''_x = \{u \in T_x M_3^4;\ (u, J_0 x) = 0,\ (u, e_5) = 0\}$. We put $M = M_3^4 \times \{1\}$. For each point $(x, 1) \in M$, let $D_{(x,1)}$ be the subspace of $T_{(x,1)}M$ defined by $D_{(x,1)} = \{(u, 0) \in T_{(x,1)}M; u \in D''_x\}$. Then we may easily observe that M is a totally umbilical generic submanifold of $\bar{M}$ with holomorphic distribution D which is not a CR-submanifold of $\bar{M}$ and is not totally geodesic in $\bar{M}$.

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