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A TYCHONOFF NON-NORMAL SPACE

V. TZANNES
Department of Mathematics
University of Patras
Patras 26110, Greece

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ABSTRACT. A Tychonoff non–normal space is constructed which can be used for the construction of a regular space on which every weakly continuous (hence every θ–continuous or η–continuous) map into a given space is constant.

KEY WORDS AND PHRASES. Tychonoff, non–normal, weakly, θ–,η– continuous maps.
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1. INTRODUCTION.

We construct for every Hausdorff space R a Tychonoff non–normal space S such that if f is a weakly continuous map of S into R then there exist two closed subsets $K', L', K' \cap L' = \emptyset$ such that $f(K') = f(L') = \{r\}$, $r \in R$. Therefore, applying the method of Jones [1], we can first construct a regular space containing two points $-\infty$, $+\infty$ such that $f(-\infty) = f(+\infty)$, for every weakly continuous map f of this space into R and then, applying the method of Iliadis and Tzannes [2], a regular space on which every weakly continuous (hence every θ–continuous or η–continuous (Dickman, Porter and Rubin [3])) map into R is constant. The construction of S is a modification of the space $T_1(R)$ in Iliadis and Tzannes [2]. For regular spaces on which every continuous map into a given space is constant see also Armentrout [4], Brandenburg and Mysior [5], van Douwen [6], Herrlich [7], Hewitt [8], Tzannes [9] and Jounglove [10]. A map $f : X \to Y$, where X, Y are topological spaces is called 1) weakly continuous if for every $x \in X$ and U open neighbourhood of $f(x)$ there exists an open neighbourhood V of x, such that $f(V) \subseteq \text{Cl}U$, 2) θ–continuous if for every $x \in X$ and open neighbourhood U of $f(x)$, there is an open neighbourhood V of x such that $f(\text{Cl}V) \subseteq \text{Cl}U$ 3) η–continuous if for every regular–open sets U, V of Y,

$$(i)\ f^{-1}(U) \subseteq \text{IntCl}f^{-1}(U)$$

$$(ii)\ \text{IntCl}f^{-1}(U \cap V) \subseteq \text{IntCl}f^{-1}(U) \cap \text{IntCl}f^{-1}(V).$$

Every η–continuous is θ–continuous (Dickman, Porter and Rubin [3, Proposition 3.3. (c)]) and every θ–continuous is obviously weakly continuous.

We denote 1) by $|X|$ the cardinality, of X, 2) by $\psi(X) = \sup\{\psi(X, x) : x \in X\}$ the pseudocharacter of X, where $\psi(X, x)$ is the pseudocharacter of X at x, that is the minimal cardinality of pseudobases of x. (The set U_α consisting of open neighbourhoods of x, is called a pseudobasis if $\cap U_\alpha = \{x\}$), 3) by $\psi^+(X)$ the smallest cardinal number greater than $\psi(X)$.

2. THE SPACE S.

Let R be a Hausdorff space and K, L two uncountable sets such that $|K| = |L| = \aleph > |R|$.

For every $k_i \in K$ (resp. $l_i \in L$) we consider an uncountable set K_i (resp. L_i) and a set M such that $|K_i| = |L_i| = |M| \geq \psi^+(R)$. On the set $S = M \bigcup KU \bigcup K_i \cup LU \bigcup L_i$ we define the following topology: Every point belonging to K_i, L_i is isolated. For every $k_i \in K$ (resp. $l_i \in L$) a basis of open neighbourhoods are the sets $O(k_i) = \{k_i\} \cup C_i$ (resp. $O(l_i) = \{l_i\} \cup D_i$), where C_i, D_i consist of all but finite number of elements of K_i, L_i, respectively. For every point $m \in M$ a basis of open neighbourhoods are the sets $O(m) = \{m\} \cup P \cup Q$, where P, Q contain all but finite number of elements of the sets $\{h_i(m) : i \in I\}, \{g_i(m) : i \in I\}$, respectively, where I is an index set, $|I| = \aleph$ and h_i, g_i are one-to-one maps of M onto K_i, L_i, respectively.

One can show that the space S is Tychonoff and non-normal.

Let f be a weakly continuous map of S into R Since $|K| > |R|$, it follows that for some $r_1 \in R$ there exists $K' \subseteq K$ such that $|K'| = |K|$ and $f(K') = \{r_1\}$. Let $\{k_n : n = 1,2,...\}$ be a countable subset of K'. Since for every open neighbourhood U of r_1 the set $f^{-1}(\mathrm{Cl}U)$ contains an open neighbourhood of $k_n, n = 1,2,...$, it follows that $|K_n \setminus f^{-1}(r_1)| \leq \psi(R, r_1)$. Consequently, if h_n is the one-to-one map of M onto K_n then $|h_n^{-1}(K_n \setminus f^{-1}(r_1))| \leq \psi(R, r_1)$ and hence $|\bigcup_{n=1}^{\infty} h_n^{-1}(K_n \setminus f^{-1}(r_1))| \leq \psi(R, r_1)$. Repeating all the above for the set L we have that for some $r_2 \in R$ there exist $L' \subseteq L, |L'| = |L|, f(L') = \{r_2\}$ and a countable subset $\{l_n : n = 1,2,...\} \subseteq L'$ such that if V is an open neighbourhood of r_2 then $|L_n \setminus f^{-1}(r_2)| \leq \psi(R, r_2)$ and hence $|\bigcup_{n=1}^{\infty} g_n^{-1}(L_n \setminus f^{-1}(r_2))| \leq \psi(R, r_2)$. Therefore if $M' = \bigcup_{n=1}^{\infty}(h_n^{-1}(K_n \setminus f^{-1}(r_1)) \cup g_n^{-1}(L_n \setminus f^{-1}(r_2)))$ then $M \setminus M' \neq \emptyset$. Let $m \in M \setminus M'$ and $\mathrm{Cl}W$ be a closed neighbourhood of $f(m)$ such that $r_1, r_2 \notin \mathrm{Cl}W$. There exists an open neighbourhood $O(m)$ of m such that $f(O(m)) \subseteq \mathrm{Cl}W$, while for every $n = 1,2,..., h_n(m) \in f^{-1}(r_1)$, $g_n(m) \in f^{-1}(r_2)$ which imply that $f(m) = r_1 = r_2$.

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