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THE FREE A-RING IS A GRADED A-RING

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ABSTRACT. In this paper, we define the free A-ring over K on a set X, categorically, and parallel some results from the theory of free algebras. We show that the free A-ring over K on X, denoted by $A_K\{X\}$, is graded.

KEY WORDS AND PHRASES. A-ring, free A-ring.
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DEFINITION 1. Let A be a K-algebra. **An A-ring over K** is a K-algebra B together with a K-algebra homomorphism $f_B: A \to B$. An **A-ring homomorphism** between A-rings B and C is a K-algebra homomorphism $g: B \to C$ such that $gf_B = f_C$.

DEFINITION 2. Let X be a set. The **free A-ring**, $A_K\{X\}$, over K on X is an A-ring containing X such that for every A-ring B and function $f: X \to B$ there is a unique A-ring homomorphism $f_*: A_K\{X\} \to B$ that extends f. $A_K\{X\}$ is the free object on X in the category of A-rings over K.

Just from the definitions of the terms involved it is easy to see that $A_K\{X\} = A^*_K K < X >$, the coproduct of A and $K < X >$, the free K-algebra on X. Also $A_K\{X,Y\} = (A_K\{X\})_K\{Y\}$. To see this recall that $K < X, Y > = K < X >^*_K K < Y >$ so that $A_K\{X,Y\} \simeq A^*_K K < X,Y > \simeq A^*_K (K < X >^*_K K < Y >) \simeq (A^*_K K < X >)^*_K K < Y > \simeq (A_K\{X\})_K\{Y\}$. Also using a categorical argument with just the definitions of the terms involved it is easy to prove that $A_K\{X,Y\} \simeq A_K\{X\}^*_A A_K\{Y\}$.

In Cartan and Eilenberg [8, p. 146] the term free A-ring on a set X is used. The terminology may be some what misleading since this notion of a free A-ring is not the free object on X in the category of A-rings. The free A-ring discussed there is essentially $A \otimes_Z Z < X >$.

Let us call an A-ring that is graded as a K-algebra a **graded A-ring** in case A is homogeneous of degree 0, i.e., the image of A is contained in the homogeneous component of degree 0.

We define the **tensor A-ring of an A-bimodule M over K** to be the A-bimodule $T_A(M) = A \oplus M \oplus (M \otimes_A M) \oplus \cdots \oplus (\otimes^n M) \oplus \cdots$. $T_A(M)$ is made into a ring in the same fashion as the tensor algebra. $T_A(M)$ is a K-algebra since A is a K-algebra and M is an A-bimodule over K. A is a sub-K-algebra of $T_A(M)$ and thus $T_A(M)$ is canonically an A-ring. Also M is a sub-A-bimodule. $T_A(M)$ is by construction a graded A-ring. As before we adopt the notation $A = \otimes^0 M$ and $M_n = \otimes^n M$. As in the case of the tensor algebra, $T_A(M)$ satisfies the following universal mapping property:

THEOREM 1. For all A-rings B and A-bimodule homomorphism $f: M \to B$, there is a unique A-ring homomorphism $f_*: T_A(M) \to B$ that extends f.

PROOF. Since B is an A-ring we have the canonical K-algebra homomorphism from A to B. Denote this map by f_o. Denote f by f_1. Let $f_n: \otimes_A^n M \to B$ be that unique A-bimodule homomorphism such that $f_n(m_1 \otimes m_2 \otimes \cdots \otimes m_n) = f_1(m_1) f(m_2) \cdots f_n(m_n)$. The f_n's induce an A-bimodule homomorphism $f_*: T_A(M) \to B$ by $f_*(\Sigma z_p) = \Sigma f_p(z_p)$. Note that $f_{r+s}(z_r w_s) = f_r(z_r) f_s(w_s)$ since if $z_r = \sum_i m_{i1} \otimes \cdots \otimes_{ir}$ and $w_s = \sum_j m'_{j1} \otimes \cdots \otimes m'_{js}$ then

$$
\begin{aligned}
f_{r+s}(z_r w_s) &= f_{r+s}\,(\sum_{i,j} m_{i1} \otimes \cdots m_{ir} \otimes m'_{j1} \otimes \cdots \otimes m'_{js}) \\
&= \sum_{i,j} f_1(m_{i1}) \cdots f_1(m_{ir}) f_1(m'_{j1}) \cdots f_1(m'_{js}) \\
&= f_r(\sum_i m_{i1} \otimes \cdots \otimes m_{ir}) f_s(\sum_j m'_{j1} \otimes \cdots \otimes m'_{js})
\end{aligned}
$$

We now show that f_* is a ring homomorphism. $f_*(\sum_p z_p \sum_q w_q)$

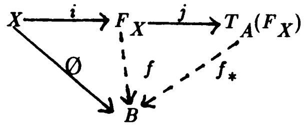
$$
\begin{aligned}
&= f_*(\sum_{p,q} z_p w_q) = f_*(z_o w_o + (z_o w_o + (z_o w_1 + z_1 w_o) + \cdots) \\
&= f_o(z_o w_o) + f_1(z_o w_1 + z_1 w_o) + \cdots \\
&= f_o(z_o) f_o(w_o) + f_o(z_o) f_1(w_1) + f_1(z_1) f_o(w_o) + \cdots \\
&= (f_o(z_o) + \cdots)(f_o(w_o) + f_1(w_1) + \cdots) \\
&= f_*(\sum_p z_p)\ (f_*\ (\sum_q w_q).
\end{aligned}
$$

$f_{*|A} = f_o$ so f_* is an A-ring homomorphism and $f_{*|M} = f_1$. If $g: T_A(M) \to B$ is another A-ring homomorphism such that $g|_M = f_1$ then since $m_1 \otimes \cdots \otimes m_n \in M_n$ is $m_1 m_2 \cdots m_n, m_i \in M$ then $g(m_1 \otimes \cdots \otimes m_n) = g(m_1 \cdots m_n) = g(m_1) \cdots g(m_n) = f_1(m_1) \cdots f_1(m_n) = f_n(m_1 \otimes \cdots \otimes m_n)$ so that since f_n is unique with respect to this property we have $g \mid M_n = f_n$ for each $n = 0, 1, \cdots$, but this means $g = f_*$.

The free K-algebra is an example of a graded K-algebra. The following theorem shows that the free A-ring is a graded A-ring.

THEOREM 2. $A_K\{X\}$ is the tensor A-ring $T_A(F_X)$, where F_X is the free A-bimodule over K on X.

PROOF. Consider the following diagram:



where i, j are inclusion maps, B is an arbitrary A-ring and $\emptyset$ is any function from X to B. Since B is an A-ring then B is an A-bimodule over K so there is a unique A-bimodule homomorphism $f: F_X \to B$ that extends $\emptyset$. By the universal property of $T_A(F_X)$ given by Theorem 1 we have a unique A-ring homomorphism $f_*: T_A(F_X) \to B$ that extends f. Thus f_* extends $\emptyset$ and if $g: T_A(F_X) \to B$ is any other A-ring homomorphism that extends $\emptyset$ then $g \mid F_X$ is an A-bimodule homomorphism from F_X to B that extends $\emptyset$ so that $g \mid F_X = f$. But f_* is unique with respect to extending f so we must have $g = f_*$.

Because of this theorem $A_K\{X\}$ could also be called the **polynomial K-algebra over A in the noncommuting indeterminates X**, where the scalars $a \in A$ do not necessarily commute with the indeterminates.

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