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THE CAUCHY PROBLEM OF THE ONE DIMENSIONAL SCHRÖDINGER EQUATION WITH NON-LOCAL POTENTIALS

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ABSTRACT. For a large class of operators A, not necessarily local, it is proved that the Cauchy problem of the Schrödinger equation:

$$-\frac{d^2f(z)}{dz^2}+Af(z)=s^2f(z),\quad f(0)=0,\quad f'(0)=1$$

possesses a unique solution in the Hilbert ($H_2(\Delta)$) and Banach ($H_1(\Delta)$) spaces of analytic functions in the unit disc $\Delta=\{z: |z|<1\}$.

KEY WORDS AND PHRASES. Cauchy problem, Schrödinger equation, Hardy-Lebesque spaces.
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1. INTRODUCTION.

Let $C[0,\pi]$ be the Banach space of continuous functions on the interval $[0,\pi]$. The norm of an element $f(x)$ of that space is defined by

$$\|f\| = \sup_{x\in[0,\pi]} |f(x)|.$$

Assume that A is a linear bounded operator on $[0,\pi]$ not necessarily local, i.e., A need not be the multiplication operator by a continuous function $a(x)$. It may, for instance, be an integral operator on $C[0,\pi]$. It is known that the Schrödinger equation:

$$-\frac{d^2f}{dx^2}+Af(x)=s^2f(x) \tag{1.1}$$

possesses a unique solution in $C[0,\pi]$ satisfying the initial conditions:

$$f(0)=0,\qquad f'(0)=1 \tag{1.2}$$

provided that $|s|>c_o\|A\|, c_o=\max_{o<\alpha<1}\frac{1-\cos\alpha\pi}{\alpha}$ [1].

Also it is known [1] that the solution is bounded for every s in the region:

$$G=\{s: |s|\geq \alpha c_o\|A\|,\quad \alpha>1\}.$$

The purpose of this paper is to prove similar results for the initial valued problem [(1.1),(1.2)] in the Hardy spaces $H_2(\Delta)$ and $H_1(\Delta)$. These are the spaces of analytic functions: $f(z)=\sum_{n=1}^{\infty}\alpha_n z^{n-1}$ in the unit disk $\Delta=\{z: |z|<1\}$, which satisfy respectively the conditions:

$\sum_{n=1}^{\infty}|\alpha_n|^2<\infty$ and $\sum_{n=1}^{\infty}|\alpha_n|<\infty$ or equivalently the conditions:

$$\sup_{o<r<1}\int_0^{2\pi}|f(re^{i\vartheta})|^2 d\vartheta<\infty \text{ and } \sup_{o<r<1}\int_0^{2\pi}|f(re^{i\vartheta})|\,d\vartheta<\infty,$$

for $re^{i\vartheta}=z$.

2. REDUCTION OF THE SCHRÖDINGER EQUATION.

$$-\frac{d^2f(z)}{dz^2}+Af(z)=s^2f(z) \tag{2.1}$$

in $H_2(\Delta)(H_1(\Delta))$ to an abstract operator form. (We follow the method prescribed in [2] and [3]).

Let H denote an abstract separable Hilbert space with an orthogonal basis $\{e_n\}_1^{\infty}$ and let V be the unilateral shift operator on H, i.e.,

$$V{:}Ve_n=e_{n+1},\ n=1,2,\cdots,\ V^*{:}V^*e_n=e_{n-1},\ n\neq 1,\ V^*e_1=0$$

is the adjoint operator of V.

Every function $f(z)=\sum_{n=1}^{\infty}\alpha_n z^{n-1}$ in $H_2(\Delta)$ can be represented as follows: $f(z)=(f_z,f)$, where $(\cdot,\cdot)$ means the scalar product in H and $f_z=\sum_{n=1}^{\infty}z^{n-1}e_n$, $|z|<1$ are the eigenelements of V^*.

The space H_1 is the Banach space which consists of those elements $f=\sum_{n=1}^{\infty}\bar{\alpha}_n e_n$, in H, (overbar means complex conjugate), that satisfy the condition $\sum_{n=1}^{\infty}|(f,e_n)|<\infty$. This space under the isomorphism $f(z)=(f_z,f)$ is isomorphic to $H_1(\Delta)$.

The norm in H_1 is denoted by: $\|f\|_1=\sum_{n=1}^{\infty}|(f,e_n)|$. To any open set or dense linear manifold E in $H(H_1)$ corresponds an open set or dense linear manifold $\bar{E}$ in $H_2(\Delta)(H_1(\Delta))$. Suppose that A is a mapping in $H_2(\Delta)(H_1(\Delta))$ and $\bar{A}$ is a mapping in $H(H_1)$. Then if the relation $Af(z)=(f_z,\bar{A}f)$ holds $\forall f\in E$, we call $\bar{A}$ the abstract form of A. For example if A is the differential operator $\frac{d^2}{dz^2}$ in $H_2(\Delta)$, i.e., $Af(z)=\frac{d^2f(z)}{dz^2}$, then $\bar{A}=(C_oV^*)^2=C_o(C_o+I)V^{*2}$, where C_o is the diagonal operator $C_oe_n=ne_n, n=1,2,\cdots$ (see for details in [2] and [3]).

Every bounded operator on $H_2(\Delta)(H)$ is defined on $H_1(\Delta)(H_1)$ and maps, in general, elements of $H_1(\Delta)(H_1)$ into $H_2(\Delta)(H)$.

The following properties follow easily:

(i) H_1 is invariant under the operators V,V^* and $\|V\|_1=\|V^*\|_1=1$, where $\|A\|_1$ means the norm of an operator on H_1.

(ii) H_1 is invariant under every bounded diagonal operator $De_n=d_ne_n, n=1,2,\cdots$ on H and $\|D\|_1=\|D\|=\sup_n|d_n|$.

(iii) For every element $f(z)=\sum_{n=1}^{\infty}\alpha_n z^{n-1}$ in $H_1(\Delta)$ the uniform limit of the sequence $\sum_{i=1}^{n}\bar{\alpha}_iV^{i-1}$, i.e., $\lim_{n\to\infty}\sum_{i=1}^{n}\bar{\alpha}_iV^{i-1}$ exists and defines a bounded operator $f^*(V)=\bar{\alpha}_1+\bar{\alpha}_2V+\bar{\alpha}_3V^2+\cdots$ on H_1. Moreover $\|f^*(V)\|_1=\|f\|_1$.

(iv) The null space of V^{*k} in H belongs to H_1.

Now we write equation (2.1) in the form:

$$\frac{d^2f}{dz^2}+s^2f(z)-Af(z)=0. \tag{2.2}$$

The abstract form of equation (2.2) is the following:

$$((C_oV^*)^2-\bar{A}_1)f=0 \tag{2.3}$$

or

$$(V^{*2}-B_1\bar{A}_1)f=0, \tag{2.4}$$

where $\overline{A}_1 = \overline{A} - s^2 I$ and B_1 is the diagonal operator on $H: B_1 e_n = \frac{1}{n(n+1)} e_n$, $n = 1, 2, \cdots$.

This means that equation (2.2) has a solution in $H_2(\Delta)(H_1(\Delta))$ satisfying the conditions $f(0) = 0, f'(0) = 1$ iff equation (2.4) has a solution in $H(H_1)$ satisfying the conditions:

$$(f, e_1) = 0, \qquad (f, e_2) = 1. \tag{2.5}$$

Note that H_1 is imbedded in H in the sense that f in H_1 implies f in H and $\| f \| \leq \| f \|_1$.

3. SOLUTION OF THE CAUCHY PROBLEM [(2.4), (2.5)] IN H AND H_1.

THEOREM 1. The equation $(V^* - B_1 \overline{A}_1) f = 0$ has at least one solution in H which satisfies the condition $(f, e_1) = 0$.

PROOF. Set $f = Vg$, then $(Vg, e_1) = (g, V^* e_1) = (g, 0) = 0$. Also $(V^{*2} - B_1 \overline{A}_1)(Vg) = 0$ implies $V^*(I - VB_1 \overline{A}_1 V) g = 0$.

Thus $(I - VB_1 \overline{A}_1 V) g = c e_1$.

Now since B_1 is compact, V and $\overline{A}$ bounded the operator $VB_1 \overline{A}_1 V$ is compact and the Fredholm alternative implies that either: $(I - VB_1 \overline{A}_1 V) g = 0$ for $g \neq 0$ or $(I - VB_1 \overline{A}_1 V)^{-1}$ exists and it is bounded.

In the first case $g \neq 0$ is a solution of equation (2.4). In the second case we have $g = c(I - VB_1 \overline{A}_1 V)^{-1} e_1 \neq 0$ for $c \neq 0$. □

Theorem 1 implies that the Schrödinger equation (2.1) has at least one solution in $H_2(\Delta)$ which satisfies the condition $f(0) = 0$, for every real or complex s, and every bounded linear operator A on $H_2(\Delta)$.

THEOREM 2. If $\| \overline{A}_1 \| < 2$, then equation (2.4) has a unique solution in H which satisfies the conditions (2.5).

PROOF. Set $f = e_2 + V^2 g$, then obviously $(f, e_1) = 0$ and $(f, e_2) = 1$. Also from equation (2.4) we get: $-B_1 \overline{A}_1 e_2 + Ig - B_1 \overline{A}_1 V^2 g = 0$ which implies that

$$(I - B_1 \overline{A}_1 V^2) g = B_1 \overline{A}_1 e_2. \tag{3.1}$$

(i) If $\overline{A}_1 e_2 = 0$, then e_2 is the unique solution in H which satisfies the initial conditions, since $(I - B_1 \overline{A}_1 V^2) g = 0$ implies that $g = 0$.

(ii) If $\overline{A}_1 e_2 \neq 0$, then from equation (3.1) since $\| B_1 \| = \frac{1}{2}$ and $\| \overline{A} \| < 2$, we easily get that $\| B_1 \overline{A}_1 V^2 \| < 1$. Hence the inverse of $(I - B_1 \overline{A}_1 V^2)$ exists and it is bounded on H. Therefore $g = (I - B_1 \overline{A}_1 V^2)^{-1} B_1 \overline{A}_1 e_2$, $g \neq 0$ and g is uniquely defined. □

There has been defined, in [3], a class of bounded operators on $H(H_1)$ which have the so-called "k-invariant property." • Abstract forms of local potentials of the form: $Af(z) = a(z) f(z)$ are included in this class.

The importance of such operators is due to the fact that if $\overline{A}_1$ is k-invariant on the space H_2, then the operator $A_2 = I - V^2 B_1 \overline{A}_1$ leaves invariant the space H_1 and when restricted on it, has a bounded inverse (see [3], Theorem 3.2).

DEFINITION. A bounded operator $\overline{A}$ on H is called k-invariant iff its adjoint $\overline{A}^*$ has the property: $\overline{A}^* e_i \in M_{i+k-1}$, where M_{i+k-1} is the subspace spanned by $\{e_1, e_2, \cdots e_{i+k-1}\}$, $i = 1, 2, \cdots$.

Such operators are the diagonal operators in the basis $\{e_n\}_1^\infty$, analytic functions of the shift V, algebraic combinations of the above and polynomial functions of V^* of degree less than k.

In accordance with the above definition a bounded operator A on $H_2(\Delta)(H_1(\Delta))$ is called 2-invariant iff its adjoint A^* has the property: $A^* z^i \in \{1, z, z^2, \cdots, z^i\}$, where $\{1, z, z^2, \cdots, z^i\}$, is the subspace of $H_2(\Delta)(H_1(\Delta))$ spanned by the elements $1, z, z^2, \cdots, z^i$.

For example the operator:

$$A = Af(z) = af(z) + zf(z) + \tfrac{1}{z}(f(z) - f(0))$$

is a 2-invariant self adjoint operator on $H_2(\Delta)$.

THEOREM 3. The Cauchy problem:

$$-\frac{d^2f(z)}{dz^2} + Af(z) = s^2 f(z) \tag{3.2}$$

$$f(0) = 0, \qquad f'(0) = 1, \tag{3.3}$$

where A is any 2-invariant operator on $H_1(\Delta)$, has a unique solution in $H_1(\Delta)$ for every $s \in \mathbf{C}$.

This solution is bounded for every z in the unit disc.

PROOF. The abstract form of (3.2) is:

$$(V^{*2} - B_1\overline{A} + s^2 B_1)f = 0 \tag{3.4}$$

and the conditions (3.3) are equivalent to

$$(f, e_1) = 0, \qquad (f, e_2) = 1. \tag{3.5}$$

Setting $f = e_2 + V^2 g$ which obviously satisfies the initial conditions (3.5) we get:

$$(I - B_1(\overline{A} - s^2)V^2)g = B_1(\overline{A} - s^2)e_2. \tag{3.6}$$

The operators V, V^* and B_1 leave the space H_1 invariant. The same holds for the operator $(I - B_1(\overline{A} - s^2)V^2)$, which restricted on H_1 has a bounded inverse (see [3], Theorem 3.2). Also $B_1(\overline{A} - s^2)e_2 = h \in H_1$ and the unique solution of (3.6) is given by: $g = (I - B_1(\overline{A} - s^2)V^2)^{-1}h$.

For every $f(z) = \sum_{n=1}^{\infty} \alpha_n z^{n-1} \in H_1(\Delta)$ we have: $|f(z)| \leq \sum_{n=1}^{\infty} |\alpha_n| = \|f(z)\|_{H_1(\Delta)} < \infty$, $|z| \leq 1$. This shows that the solution predicted by the theorem is bounded for $|z| \leq 1$. □

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