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COMMON STATIONARY POINTS FOR SET-VALUED MAPPINGS

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ABSTRACT. Several theorems on stationary points for set-valued mappings have obtained. These are improvements upon some earlier results due to Fisher.

KEY WORDS AND PHRASES. Generalized Hausdorff distance, nearly-densifying mappings, orbit, common stationary points.
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1. INTRODUCTION AND PRELIMINARIES.

In this paper, we prove several common stationary point theorems for four set-valued mappings, which are improvements upon some earlier results obtained by Fisher [1], [2], [3].

Let (X,d) be a metric space and $CL(X)$ be the class of all nonempty closed subset of X. For $x \in X$ and $A \subseteq X$, let $D(x,A) = inf\{d(x,y): y \in A\}$.

DEFINITION 1.1. For $A, B \in CL(X)$, define

$$H(A,B) = \begin{cases} max\{sup_{x \in A} D(x,B),\ sup_{y \in B} D(A,y)\}, & \text{if it exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

Then H is called the ***generalized Hausdorff distance function*** for the class $CL(X)$ induced by the metric d.

DEFINITION 1.2. For $A, B \in CL(X)$, define $h: CL(X) \times CL(X) \to R^+$ by

$$h(A,B) = \begin{cases} sup\{d(x,y) : x \in A, y \in B\}, & \text{if it exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

DEFINITION 1.3. A set-valued mapping $S: X \to CL(X)$ is said to be ***nearly-densifying*** if $\alpha(S(A)) < \alpha(A)$ for any bounded and S-invariant subset of X with $\alpha(A) > 0$, where α is the Kuratowski's measure of non-compactness.

DEFINITION 1.4. Let $F, G, S, T : X \to CL(X)$ be set-valued mappings. For some $x \in X$, define the ***orbit*** $O(x)$ of x by

$$O(x) = \{y \in X: y = x \text{ or } y = f(x) \text{ for some } f \in \mathfrak{T}\},$$

$\mathfrak{T}$ being the subsemigroup generated by F,G,S and T in the semigroup of all self-mappings on X with composition operation.

DEFINITION 1.5. A point z is said to be a ***common stationary point*** of set-valued mappings F and G if $Fz = \{z\} = Gz$.

2. THE MAIN RESULTS.

Throughout this paper, for any set-valued mapping $S: X \to CL(X)$, we assume that all the powers of S map X into $CL(X)$. First of all, we prove the following crucial result to be used in the sequel.

LEMMA 2.1. Let (X,d) be a compact metric space and $S: X \to CL(X)$ be a set-valued mapping such that S^i is continuous with respect to the generalized Hausdorff distance function H for some positive integer i. If $A = \cap_{k=1}^{\infty} S^{ki}(X)$, then $S(A) = A$.

PROOF. Clearly, $S^{(k+1)i}(X) \subset S^{ki}(X)$ for $k = 1,2,\cdots$. Also, $x \in X$ implies

$$Sx \subseteq A. \tag{1.1}$$

Let $y \in A$. Then $y \in S^{(k+1)i}(X)$ for $k = 1,2,\cdots$, and so there exists $x_k \in S^{ki}(X)$ such that $y = S^i x_k$ for $k = 1,2,\cdots$. Since X is compact, there exists a convergent subsequence $\{x_{k_l}\}$ of $\{x_k\}$ with the limit z. Further, since $\{x_j, x_{j+1},\cdots\} \subseteq S^{ji}(X)$ for $j = 1,2,\cdots$, we have $z \in A$. Also, we have

$$D(y, S^i z) \leq D(y, S^i x_{k_\ell}) + H(S^i x_{k_\ell}, S^i z).$$

Letting $l \to 0$, we get $y \in S^i z$. Hence there exist $x_i, x_{i-1}, \cdots, x_2 \in X$ such that $y \in Sx_i$, $x_i \in Sx_{i-1}, \cdots, x_3 \in Sx_2$, and $x_2 \in Sz$. By (1.1), since $z \in A$, it follows that $Sz \subseteq A$ and so $x_2 \in A$. A repeated application of (1.1) yields that $x_i \in A$. Therefore, we have $y \in Sx$ for some $x \in A$. Thus, $A \subseteq S(A)$. From this and (1.1), we conclude that $S(A) = A$. This completes the proof.

Now, we are in a position to present our main results. We denote

$$M(x,y,F^p,G^q,S^s,T^t) = max\{h(S^s x, T^t y), h(S^s x, F^p x), h(T^t y, G^q y),\ h(S^s x, G^q y), h(T^t y, F^p x)\}$$

and

$$m(x,y,F^p,G^q,S^s,T^t) = max\{h(S^s x, T^t y), h(S^s x, G^q y), h(T^t y, F^p x)\},$$

where p,q,s and t are positive fixed integers.

THEOREM 2.1. Let (X,d) be a complete metric space and $F,G,S,T: X \to CL(X)$ be set-valued mappings such that

(2.1) F,G,S,T and $(FG)^i$ are continuous with respect to the distance function H for some positive integer i. Also, F,G,S and T are nearly-densifying,

(2.2) for some $x_o \in X$, the orbit $O(x_o)$ is bounded,

(2.3) $H(F^p x, G^q y) < M(x,y,F^p,G^q,S^s,T^t)$,

(2.4) $FG = GF, (FG)^i S^s = S^s (FG)^i$ and $(FG)^i T^t = T^t (FG)^i$.

Then F,G,S and T have a unique common stationary point z in X.

PROOF. Putting $A = O(x_o)$, we have clearly $I(A) = A$ for $I \in \{F,G,S,T\}$. Also, the continuity of set-valued mappings F,G,S and T yields that $I(\bar{A}) \subseteq \bar{A}$ for $I \in \{F,G,S,T\}$. Further, we have $A = \{x_o\} \cup F(A) \cup G(A) \cup S(A) \cup T(A)$. Thus, $\alpha(A) = max\{\alpha(x_o), \alpha(F(A)), \alpha(G(A)), \alpha(S(A)), \alpha(T(A))\}$ and also $\bar{A}$ is compact. Now, define $B = \cap_{n=1}^{\infty} (FG)^{in}(\bar{A})$. Then B is compact. By Lemma 2.1, $(FG)(B) = B$ and the condition (2.4) ensures that $F(B) = B = G(B)$, $S^s(B) \subseteq B$ and $T^t(B) \subseteq B$. Since B is compact, there exist $x_1, x_2 \in B$ such that $d(x_1,x_2) = sup\{d(x,y) : x,y \in B\} = \delta(B)$, say. Also, there exist $w_1, w_2 \in B$ such that $x_1 \in F^p w_1$ and $x_2 \in G^q w_2$. Suppose that $\delta(B) > 0$. Then, by (2.3), we have

$$\delta(B) = d(x_1, x_2) \le H(F^p w_1, G^q w_2)$$
$$< M(w_1, w_2, F^p, G^q, S^s, T^t)$$
$$\le \delta(B),$$

which is a contradiction. Thus, $\delta(B) = 0$ and hence $B = \{z\}$, say. Therefore, z is a common stationary point of F, G, S and T. The uniqueness of z follows from condition (2.3). This completes the proof.

THEOREM 2.2. Let (X,d) be a compact metric space and $F,G,S,T: X \to CL(X)$ be set-valued mappings such that

(2.5) $(FG)^i$ is continuous for some positive integer i,

(2.6) $H(F^p x, G^q y) < M(x, y, F^p, G^q, S^s, T^t)$ whenever the left-hand side is positive,

(2.7) $FG = GF, (FG)^i S^s = S^s (FG)^i$ and $(FG)^i T^t = T^t (FG)^i$.

Then F, G, S and T have a unique common stationary point z in X. Further, z is the unique common stationary point of F and G.

PROOF. If we put $B = \cap_{n=1}^{\infty} (FG)^{i_n}(X)$, as in the proof of Theorem 2.1, we have $B = \{z\}$ and z is a unique common stationary point of F, G, S and T. Since any common stationary point of F and G is a point of $B = \{z\}$, it follows that z is the unique common stationary point of F and G. This completes the proof.

REMARK. Theorem 2 of Fisher [2] and theorems in Fisher [3] follow as corollaries of our Theorem 2.2. In fact, our theorem can be regarded as an improvement over the above theorems due to Fisher.

THEOREM 2.3. Let (X,d) be a complete metric space and $F,G,S,T: X \to CL(X)$ be set-valued mappings such that

(2.8) F, G, S, T, F^i and G^j are continuous with respect to the distance function H for some positive integers i and j. Also, F, G, S and T are nearly-densifying,

(2.9) for some $x_o \in X$, the orbit $O(x_o)$ is bounded,

(2.10) $H(F^p x, G^q y) < m(x, y, F^p, G^q, S^s, T^t)$ whenever the left-hand side is positive,

(2.11) $S^s F^i = F^i S^s$ and $T^t G^q = G^q T^t$.

Then F, G, S and T have a unique common stationary point z in X.

PROOF. Let $A = O(x_o)$. Then as in the proof of Theorem 2.1, $\bar{A}$ is compact. If we define

$$B = \cap_{n=1}^{\infty} F^{i_n}(A) \text{ and } K = \cap_{n=1}^{\infty} G^{j_n}(A),$$

by Lemma 2.1, $F(B) = B$ and $G(K) = K$. Also, it follows that B and K are compact subsets of X. By the condition (2.11), also we have $S^s(B) \subseteq B$ and $T^t(K) \subseteq K$. Then, there exist $x_1, w_1 \in B$ and $y_1, y_2 \in K$ such that

$$d(x_1, y_1) = sup\{d(x,y): x \in B, y \in K\} = \delta(B,K), \text{ say},$$

with $x_1 \in F^p w_1$ and $y \in G^q w_2$. Suppose that $\delta(B,K) > 0$. Then, by the condition (2.10), we have

$$\delta(B,K) = d(x_1, y_1)$$
$$\le H(F^p w_1, G^q w_2)$$
$$< m(w_1, w_2, F^p, G^q, S^s, T^t)$$
$$\le \delta(B,K),$$

which is a contradiction. Therefore, $\delta(B,K)=0$ and $B=K=\{z\}$. Thus z is a common stationary point of F,G,S and T. The uniqueness of z follows easily from the condition (2.10). This completes the proof.

THEOREM 2.4. Let (X,d) be a compact metric space and $F,G,S,T: X \to CL(X)$ be set-valued mappings such that

(2.12) F^i and G^j are continuous with respect to the distance function H for some positive integers i and j,

(2.13) $H(F^p x, G^q y) < m(x,y,F^p,G^q,S^s,T^t)$ whenever the left-hand side is positive,

(2.14) $F^i S^s = S^s F^i$ and $G^q T^t = T^t G^q$.

Then F,G,S and T have a unique common stationary point z in X. Further, z is the unique common stationary point of the pairs F,S and G,T. Also, z is the unique common stationary point of F and G.

PROOF. Let $B = \cap_{n=1}^{\infty} F^{in}(X)$ and $K = \cap_{n=1}^{\infty} G^{jn}(X)$. Then as in the proof of Theorem 2.3, we get $B=K=\{z\}$ and z is a unique common stationary point of F,G,S and T. Since any stationary point of F is a point of $B=\{z\}$ and any stationary point of G is a point of $K=\{z\}$, it follows that z is the unique stationary point of F as well as of G. This completes the proof.

REMARK. Theorem 5 of Fisher [1] follows as a corollary of our Theorem 2.3.

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