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FLOWS FOR CHOSEN VORTICITY FUNCTIONS— EXACT SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

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ABSTRACT. Solutions are obtained for the equations of the motion of the steady incompressible viscous planar generalized Beltrami flows when the vorticity distribution is given by $\nabla^2\psi = \psi + f(x,y)$ for three chosen forms of $f(x,y)$.

KEY WORDS AND PHRASES. viscous flow, asymptotic suction profile, Beltrami flow.

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1. INTRODUCTION.

Only a small number of exact solutions of the Navier-Stokes equations has been found and Chang-Yi Wang [1] has given an excellent review of these solutions. These known solutions of viscous incompressible Newtonian fluids may be classified into three types:

(i) Flows for which the non-linear inertia terms in the linear momentum equations vanish identically. Parallel flows and flows with uniform suction are examples of these flows;

(ii) flows with similarity properties such that the flow equations reduce to a set of ordinary differential equations. Stagnation point flow is an example of such flows;

(iii) flows for which the vorticity function is so chosen that the governing equation in terms of the stream function reduces to a linear equation. Taylor [2], Kampe de Feriet [3], Kovasznay [4], Wang [5] and Lin and Tobak [6] employed this approach, taking $\nabla^2\psi = K\psi$, $\nabla^2\psi = f(\psi)$. $\nabla^2\psi = y + (K^2 - 4\pi^2)\psi$, $\nabla^2\psi = A\psi + Cy$ and $\nabla^2\psi = K(\psi - Ry)$, respectively.

In this paper, we study generalized Beltrami flows when the vorticity function $\omega = -\nabla^2\psi$ is given by $\nabla^2\psi = \psi + Ay^2 + Bxy + Cx + Dy$, $\nabla^2\psi = \psi + Ay^2 + Cx + D$, $\nabla^2\psi = \psi + Cx + Dy$, where A, B, C, D are real constants.

2. BASIC EQUATIONS AND SOLUTIONS.

Steady plane incompressible viscous fluid flow, in the absence of external forces, is governed by the system:

$$\begin{aligned} &\bar{u}_{\bar{x}} + \bar{v}_{\bar{y}} = 0 \\ &\bar{u}\bar{u}_{\bar{x}} + \bar{v}\bar{u}_{\bar{y}} + \frac{1}{\rho}\bar{p}_{\bar{x}} = \mu\bar{\nabla}^2\bar{u} \\ &\bar{u}\bar{v}_{\bar{x}} + \bar{v}\bar{v}_{\bar{y}} + \frac{1}{\rho}\bar{p}_{\bar{y}} = \mu\bar{\nabla}^2\bar{v} \end{aligned} \tag{2.1}$$

where $\bar{u}(\bar{x},\bar{y})$, $\bar{v}(\bar{x},\bar{y})$ are the velocity components, $\bar{p}(\bar{x},\bar{y})$ the pressure function, ρ the constant density, μ the constant viscosity and $\bar{\nabla}^2 = \frac{\partial^2}{\partial\bar{x}^2} + \frac{\partial^2}{\partial\bar{y}^2}$ is the Laplacian operator. The vorticity function for this flow is given by

$$\bar{\omega} = \bar{v}_{\bar{x}} - \bar{u}_{\bar{y}} \tag{2.2}$$

Letting U, L to be the characteristic velocity and length respectively, we introduce the non-dimensional variables

$$x = \frac{\bar{x}}{L}, \quad y = \frac{\bar{y}}{L}, \quad u = \frac{\bar{u}}{U}, \quad v = \frac{\bar{v}}{U}, \quad \omega = \frac{L\bar{\omega}}{U}, \quad p = \frac{\bar{p}}{\rho U^2} \tag{2.3}$$

in system (2.1) and equation (2.2). We apply the integrability condition $p_{xy} = p_{yx}$ to the linear momentum equations to find that u, v, ω must satisfy the system:

$$\begin{aligned} &u_x + v_y = 0 \\ &u\omega_x + v\omega_y = \frac{1}{R}\nabla^2\omega \\ &v_x - u_y = \omega \end{aligned} \tag{2.4}$$

where $R = \frac{\rho UL}{\mu}$ is the Reynolds number.

Introducing the stream function $\psi(x, y)$ such that

$$u = \psi_y, \quad v = -\psi_x \tag{2.5}$$

in system (2.4), we find that $\psi(x, y)$ must satisfy

$$\nabla^4\psi + R\frac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)} = 0 \tag{2.6}$$

In this paper, we study flows for which the vorticity distributions take the forms

$$(a) \qquad \omega = -\nabla^2\psi = -(\psi + Ay^2 + Bxy + Cx + Dy) \tag{2.7}$$

$$(b) \qquad \omega = -\nabla^2\psi = -(\psi + Ay^2 + Cx + Dy) \tag{2.8}$$

$$(c) \qquad \omega = -\nabla^2\psi = -(\psi + Cx + Dy) \tag{2.9}$$

where A, B, C, D are real constants.

Form (a):

Substituting (2.7) in the compatibility equation (2.6), we get

$$R(2Ay + Bx + D)\psi_x - R(By + C)\psi_y + \psi + Ay^2 + Bxy + Cx + Dy + 2A = 0 \tag{2.10}$$

Employing the canonical coordinates

$$\xi = Ay^2 + Bxy + Cx + Dy, \quad \eta = y \tag{2.11}$$

where $(By + C) \neq 0$, (2.10) may be written as

$$-R(B\eta + C)\psi_\eta + \psi + \xi + 2A = 0. \tag{2.12}$$

This equation is solved to obtain

$$\psi = f(\xi)(By + D)^{\frac{1}{RB}} - (Ay^2 + Bxy + Cx + Dy + 2A) \tag{2.13}$$

where f is an arbitrary function of ξ. Introducing (2.13) into (2.7), we get

$$\begin{aligned}
&\{R^2\left[C^2(C^2 + D^2) + 2BCD\xi + B^2\xi^2\right] f''(\xi) + 2R\left[C(RAC + D) - B\xi\right] f'(\xi) \\
&+ \left[1 - RB - R^2C^2\right] f(\xi)\} + 2RC\left\{2R\left[C(AD + BC) + AB\xi\right] f''(\xi)\right. \\
&+ 2A\left[RB + 1\right] f'(\xi) - RBf(\xi)\} \eta + R\left\{2R\left[C^2(2A^2 + 3B^2) + ABCD\right.\right. \\
&+ \left. AB^2\xi\right] f''(\xi) + 2AB\left[RB + 1\right] f'(\xi) - RB^2 f(\xi)\} \eta^2 \\
&+ 4R^2BC\left\{\left[A^2 + B^2\right] f''(\xi)\right\} \eta^3 + R^2B^2\left\{\left[A^2 + B^2\right] f''(\xi)\right\} \eta^4 = 0
\end{aligned} \tag{2.14}$$

Since ξ, η are independent variables and $\{1, \eta, \eta^2, \eta^3, \eta^4\}$ is a linearly independent set, it follows that the coefficients of the various powers of η are zero. Taking the coefficients of $\eta^4, \eta^3, \eta^2, \eta$ and 1 equal to zero, we get

$$f(\xi) = c_1\xi + c_2 \tag{2.15}$$

$$2A(RB + 1)c_1 - RBc_2 - RBc_1\xi = 0 \tag{2.16}$$

where c_1, c_2 are arbitrary constants. Since $\{1, \xi\}$ is a linearly independent set, it follows from (2.16) that $2A(RB + 1)c_1 - RBc_2 = 0$, $RBc_1 = 0$ giving $c_1 = c_2 = 0$. Using $c_1 = c_2 = 0$ in (2.15), we obtain $f(\xi) = 0$.

From (10), the stream function is given by

$$\psi(x, y) = -(Ay^2 + Bxy + Cx + Dy + 2A) \tag{2.17}$$

The exact integral of this flow is

$$\begin{aligned}
&u = -(2Ay + Bx + D), \quad v = By + C, \quad \text{and} \\
&p = p_0 - \frac{1}{2}\left[B^2(x^2 + y^2) + 2(BD - 2AC)x + 2BCy\right]
\end{aligned} \tag{2.18}$$

where p_0 is an arbitrary constant.

Equation (2.17) represents an impingement of two constant-vorticity oblique flows with stagnation point

$$(x, y) = \left(\frac{2AC - BD}{B^2}, -\frac{C}{B}\right) \tag{2.19}$$

for non-zero values of A, B, C and E. The stagnation point shifts upward as B gets smaller for fixed values of A, C and E. We remark that when $A = B = -1$, $C = D = 0$, the solution (2.17) reduces to one of the flows in Wang's [1] paper.

Form (b):

Employing (2.8) in (2.6), we obtain

$$R(2Ay + D)\psi_x - RC\psi_y + \psi + Ay^2 + Cx + Dy + 2A = 0 \tag{2.20}$$

Choosing the canonical coordinates

$$\xi = Ay^2 + Cx + Dy, \quad \eta = y \tag{2.21}$$

where $C \neq 0$, (16) takes the form

$$-RC\psi_\eta + \psi + \xi + 2A = 0. \tag{2.22}$$

We solve this equation to get

$$\psi = g(\xi)\exp\left(\frac{1}{RC}y\right) - (Ay^2 + Cx + Dy + 2A) \tag{2.23}$$

where g is an arbitrary function of ξ. We substitute (2.23) into (2.8) to get

$$\begin{aligned}&\left[R^2C^4g''(\xi) + 2R^2AC^2g'(\xi) + (1 - R^2C^2g(\xi)\right] + 2RCg'(\xi)(2A\eta + D)\\&+ R^2C^2g''(\xi)(2A\eta + D)^2 = 0\end{aligned} \tag{2.24}$$

Since ξ, η are independent variables and $\{1, (2A\eta + D), (2A\eta + D)^2\}$ is a linearly independent set, it follows that

$$g''(\xi) = 0, \quad g'(\xi) = 0, \quad (1 - R^2C^2)g(\xi) = 0 \tag{2.25}$$

From $(1 - R^2C^2)g(\xi) = 0$, we get the three possibilities: $g(\xi) = 0$, $R^2C^2 \neq 1$; $R^2C^2 = 1$, $g(\xi) \neq 0$; $g(\xi) = 0$, $R^2C^2 = 1$.

The stream function (2.23) is given by

$$\psi(x,y) = \begin{cases} -(Ay^2 + Cx + Dy + 2A) & ; g = 0, \quad R^2C^2 \neq 1 \\ K\exp\left(\frac{1}{RC}y\right) - (Ay^2 + Cx + Dy + 2A); & R^2C^2 = 1, \quad g \neq 0 \\ -(Ay^2 + Cx + Dy + 2A) & ; g = 0, \quad R^2C^2 = 1 \end{cases} \tag{2.26}$$

where $g \neq 0$ implies $g = K$ (non-zero constant).

When the stream function is given by

$$\psi(x,y) = -(Ay^2 + Cx + Dy + 2A); \quad R^2C^2 = 1 \quad \text{or} \quad R^2C^2 \neq 1, \tag{2.27}$$

the exact integral for the flow is

$$u = -(2Ay + D), \quad v = C, \quad \text{and} \quad p = p_0 + 2ACx \tag{2.28}$$

where p_0 is an arbitrary constant.

The solution (2.28) may be realized on a plate situated along $y = -\frac{D}{2A}$ with uniform suction or blowing. $C > 0$ and $C < 0$, respectively, for blowing and suction at the plate.

The exact integral for the flow given by the stream function

$$\psi(x,y) = K\exp\left(\frac{1}{RC}y\right) - (Ay^2 + Cx + Dy + 2A); \quad R^2C^2 = 1 \tag{2.29}$$

is

$$u = \frac{K}{RC}\exp\left(\frac{1}{RC}y\right) - (2Ay + D), \quad v = C, \quad \text{and} \quad p = p_0 + 2ACx \tag{2.30}$$

where p_0 is an arbitrary constant.

If $K = RCD$ in (2.29) and (2.30), the velocity profile in (2.30) can be realized on a plate located along $y = 0$ with uniform suction. The velocity profile attains the form

$$u = D\exp\left(\frac{1}{RC}y\right) - (2Ay + D), \quad v = C \tag{2.31}$$

only asymptotically, and so may be regarded as the asymptotic suction profile [7]. $C > 0$ and $C < 0$ for blowing and suction at the plate, respectively.

Form (c):

Substitution of (2.8) into (2.6) yields

$$RD\psi_x - RC\psi_y + \psi + Cx + Dy = 0 \tag{2.32}$$

The canonical coordinates

$$\xi = Cx + Dy, \quad \eta = y; \quad C \neq 0 \tag{2.33}$$

are employed in (2.32) to get

$$-RC\psi_\eta + \psi + \xi = 0.$$

The solution of this equation is

$$\psi = h(\xi)\exp\left(\frac{1}{RC}y\right) - (Dx + Ey) \tag{2.34}$$

where h is an arbitrary function of ξ. We employ (2.34) in (2.9) to obtain

$$R^2C^2(C^2 + D^2)h''(\xi) + 2RCDh'(\xi) + (1 - R^2C^2)h(\xi) = 0 \tag{2.35}$$

The general solution of (2.35) is

$$h(\xi) = \begin{cases} A_1\exp(\lambda_1\xi) + A_2\exp(\lambda_2\xi) & ;R^2(C^2 + D^2) - 1 > 0 \\ (B_1 + B_2\xi)\exp\left(-\frac{RD}{C}\xi\right) & ;R^2(C^2 + D^2) - 1 = 0 \\ C_1\,\mathrm{Cos}(m\xi + C_2)\exp\left[-\frac{D}{RC(C^2 + D^2)}\xi\right] & ;R^2(C^2 + D^2) - 1 < 0 \end{cases} \tag{2.36}$$

where

$$\lambda_{1,2} = \frac{-D \pm C\sqrt{R^2(C^2+D^2)-1}}{RC(C^2+D^2)}, \quad m = \frac{\sqrt{1-R^2(C^2+D^2)}}{R(C^2+D^2)} \tag{2.37}$$

and $A_1, A_2, B_1, B_2, C_1, C_2$ are arbitrary constants.

We shall study these three possibilities separately.

(i) $R^2(C^2+D^2)-1>0$

The stream function, from (2.34) and (2.36), is

$$\psi(x,y) = A_1 \exp\left[\lambda_1 Cx + \left(\lambda_1 D + \frac{1}{RC}\right) y\right] + A_2 \exp\left[\lambda_2 Cx + \left(\lambda_2 D + \frac{1}{RC}\right) y\right] - (Cx+Dy) \tag{2.38}$$

The exact integral of this flow is

$$\begin{aligned} u &= \left(\lambda_1 D + \frac{1}{RC}\right) A_1 \exp\left[\lambda_1 Cx + \left(\lambda_1 D + \frac{1}{RC}\right) y\right] \\ &\quad + \left(\lambda_2 D + \frac{1}{RC}\right) A_2 \exp\left[\lambda_2 Cx + \left(\lambda_2 D + \frac{1}{RC}\right) y\right] - D, \\ v &= -D\left\{\lambda_1 A_1 \exp\left[\lambda_1 Cx + \left(\lambda_1 D + \frac{1}{RC}\right) y\right]\right. \\ &\quad \left. + \lambda_2 A_2 \exp\left[\lambda_2 Cx + \left(\lambda_2 D + \frac{1}{RC}\right) y\right] - 1\right\}, \end{aligned} \tag{2.39}$$

and

$$p = p_0 + 2\left[1 - \frac{1}{R^2(C^2+D^2)}\right] A_1 A_2 \exp\left[\frac{2(Dy-Ex)}{R(C^2+D^2)}\right]$$

where p_0 is an arbitrary constant and λ_1, λ_2 are given by (2.37).

This flow represents an impingement of an oblique uniform stream with an oblique rotational, divergent flow, with stagnation point

$$\begin{aligned} (x,y) = -\frac{RC}{2\sqrt{R^2(C^2+D^2)-1}} &\left(C \ln\left(-\frac{A_1}{A_2}\right)\right. \\ &- D\sqrt{R^2(C^2+D^2)-1}\,\ln\left\{\frac{-4A_1A_2[R^2(C^2+D^2)-1]}{R^2(C^2+D^2)^2}\right\}, \\ &\left. D\ln\left(-\frac{A_1}{A_2}\right) + C\sqrt{R^2(C^2+D^2)-1}\,\ln\left\{\frac{-4A_1A_2[R^2(C^2+D^2)-1]}{R^2(C^2+D^2)^2}\right\}\right) \end{aligned} \tag{2.40}$$

where A_1, A_2 are non-zero real constants and either $A_1 > 0$, $A_2 < 0$ or $A_1 < 0$, $A_2 > 0$. For fixed values of R, C and D, the stagnation point shifts upward when the absolute value of A_2 is larger than that of A_1.

If A_1 and A_2 are of the same sign, the above phenomenon does not take place, and we have a flow without a stagnation point.

(ii) $R^2(C^2+D^2)-1=0$

Using (2.36) in (2.34), the stream function is

$$\psi(x,y) = [B_1 + B_2(Cx+Dy)]\exp[R(Cy-Dx)] - (Cx+Dy) \tag{2.41}$$

This flow has the exact integral

$$\begin{aligned} u &= \{DB_2 + RC[B_1 + B_2(Cx+Dy)]\}\exp[R(Cy-Dx)] - E, \\ v &= \{-DB_2 + RD[B_1 + B_2(Cx+Dy)]\}\exp[R(Cy-Dx)] + D, \quad \text{and} \\ p &= p_0 - \frac{1}{2R^2}B_2^2 \exp[2R(Cy-Dx)] \end{aligned} \tag{2.42}$$

where p_0 is an arbitrary constant.

If B_2 is a positive real constant, this flow represents an impingement of an oblique uniform stream with an oblique rotational, divergent flow, with stagnation point

$$(x,y) = -\frac{1}{C^2+D^2}\left(\frac{CB_1}{B_2} - \frac{D}{R}\ln B_2, \frac{DB_1}{B_2} + \frac{C}{R}\ln B_2\right) \tag{2.43}$$

For fixed values of R and C, the stagnation point shifts upward if B_1 and D are of opposite signs and the absolute value of B_1 is larger than B_2.

If B_2 is a negative real constant, (2.41) represents an oblique uniform stream which abuts on an oblique rotational, convergent flow.

(iii) $R^2(C^2+D^2) - 1 < 0$

From (2.27) and (2.36), the stream function is given by

$$\psi(x,y) = C_1 \mathrm{Cos}\left[m(Cx+Dy)+C_2\right]\exp\left[\frac{Cy-Dx}{R(C^2+D^2)}\right] - (Cx+Dy) \tag{2.44}$$

The exact integral for this flow is

$$\begin{aligned} u &= \frac{C_1}{R(C^2+D^2)}\{C\mathrm{Cos}\left[m(Cx+Dy)+C_2\right] \\ &\quad -mRD(C^2+D^2)\mathrm{Sin}\left[m(Cx+Dy)+C_2\right]\}\exp\left[\frac{Cy-Dx}{R(C^2+D^2)}\right] - D, \\ v &= \frac{C_1}{R(C^2+D^2)}\{D\mathrm{Cos}\left[m(Cx+Dy)+C_2\right] \\ &\quad +mRC(C^2+D^2)\mathrm{Sin}\left[m(Cx+Dy)+C_2\right]\}\exp\left[\frac{Cy-Dx}{R(C^2+D^2)}\right] + C, \quad \text{and} \\ &p_0 + \frac{1}{2}\left[1 - \frac{1}{R^2(C^2+D^2)}\right]C_1^2\mathrm{Cos}2\left[m(Cx+Dy)+C_2\right]\exp\left[\frac{2(Cy-Dx)}{R(C^2+D^2)}\right] \end{aligned} \tag{2.45}$$

where p_0 is an arbitrary constant, and m is given by (2.37).

If $C_1 > 0$, the stagnation points for this flow are

$$\begin{aligned} (x,y) = \Bigg(&\frac{RC[(2n+1)\frac{\pi}{2} - C_2]}{\sqrt{1-R^2(C^2+D^2)}} + RD\ln\left[\frac{C_1\sqrt{1-R^2(C^2+D^2)}}{R(C^2+D^2)}\right], \\ &\frac{RD[(2n+1)\frac{\pi}{2} - C_2]}{\sqrt{1-R^2(C^2+D^2)}} - RC\ln\left[\frac{C_1\sqrt{1-R^2(C^2+D^2)}}{R(C^2+D^2)}\right]\Bigg) \end{aligned} \tag{2.46}$$

where n is an integer.

Fig. 1 shows the streamlines for $\psi(x,y) = -(Ay^2 + Bxy + Cx + Dy + 2A)$ when $A = B = C = D = 1$. Figures 2 and 3 represent the flows $\psi(x,y) = -(Ay^2 + Cx + Dy + 2A)$ and $\psi(x,y) = K\exp\left(\frac{1}{RC}y\right) - (Ay^2 + Cx + Dy + 2A)$ for $K = R = A = C = D = 1$. Figures 4 and 5 illustrate the case (c) $\left(\nabla^2\psi = \psi + Cx + Dy\right)$ when $R^2(C^2+D^2) > 1$. Figure 4 shows reversed flow. $C = D = 1$, $R = 2$, $A_1 = 50$, $A_2 = 60$ and $C = D = R = 1$, $A_1 = 1$, $A_2 = -1$, respectively, for Figures 4 and 5. The flows when $R^2(C^2+D^2) = 1$ are given in Figures 6 and 7 when $C = D = 1$, $R = \frac{1}{\sqrt{2}}$, $B_1 = 50$, $B_2 = -60$ and $C = D = 1$, $R = \frac{1}{\sqrt{2}}$, $B_1 = 0$, $B_2 = 1$. When R^2 $(C^2+D^2) < 1$. we have Figure 8 for $C = D = 1$, $R = \frac{1}{2}$, $C_1 = 5$, $C_2 = 0$.

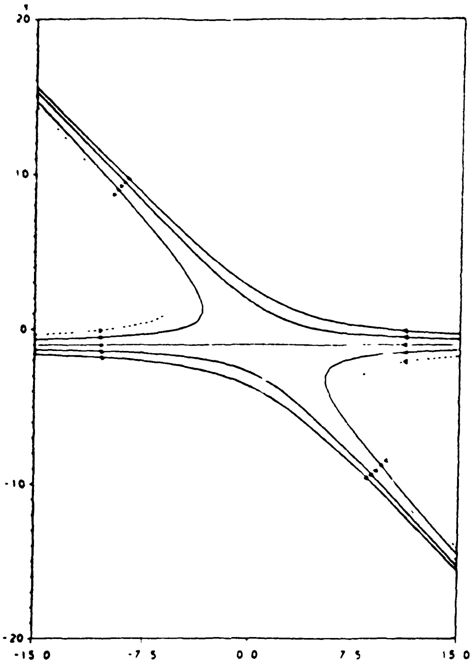

Figure 1

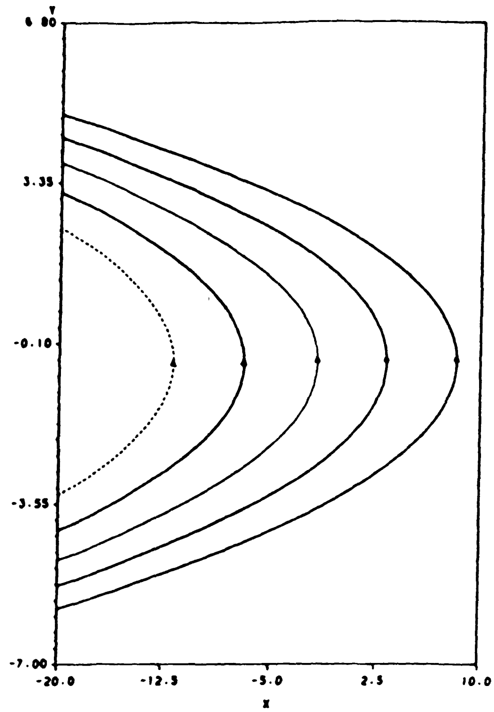

Figure 2

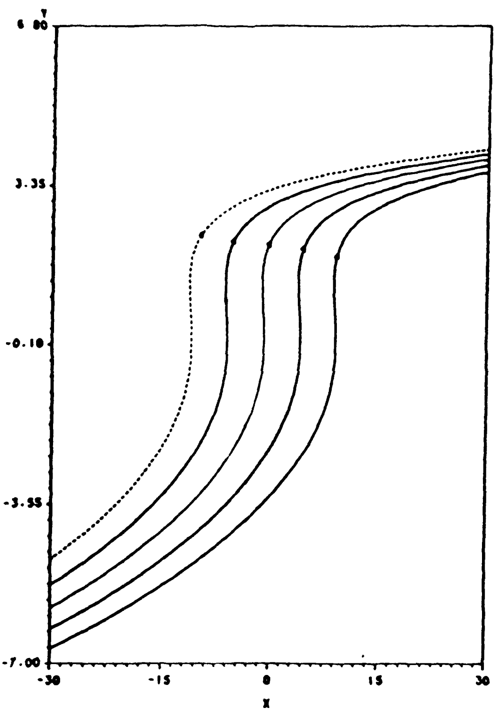

Figure 3

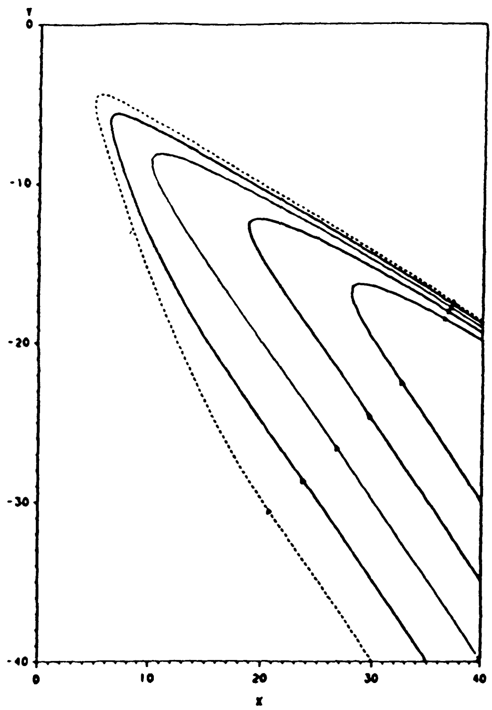

Figure 4

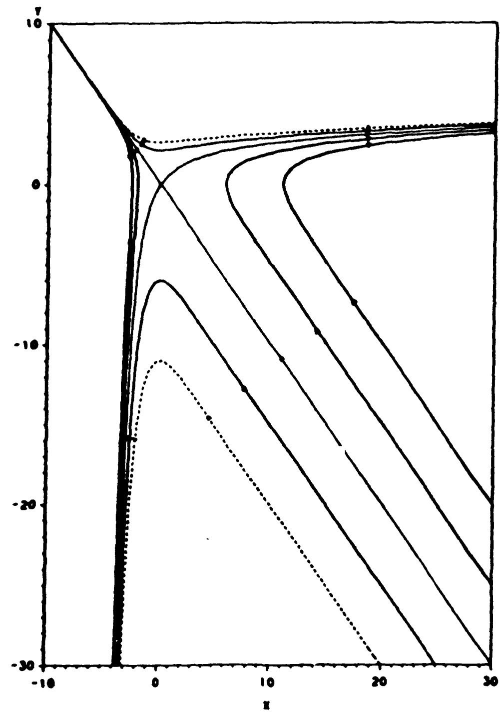

Figure 5

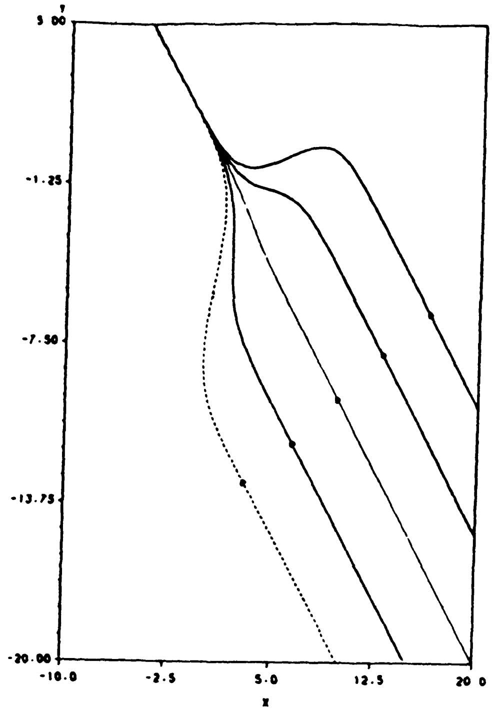

Figure 6

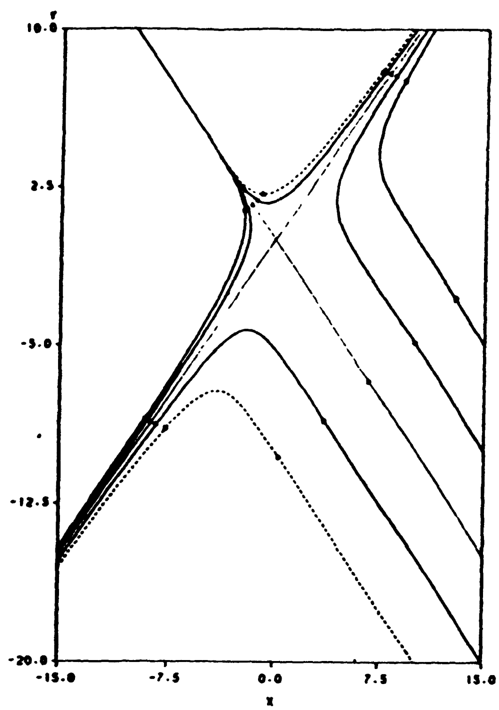

Figure 7

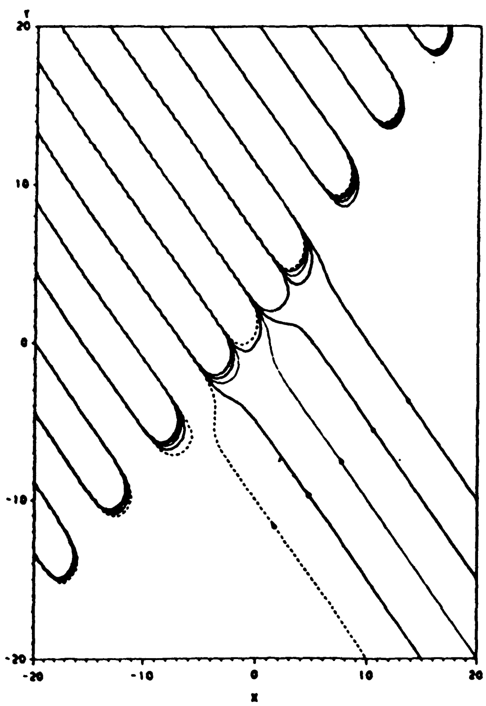

Figure 8

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