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ON THE ME-MANIFOLD IN n-*g-UFT AND ITS CONFORMAL CHANGE

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ABSTRACT. An Einstein's connection which takes the form (3.1) is called an ME-connection. A generalized n-dimensional Riemannian manifold X_n on which the differential geometric structure is imposed by a tensor field $^*g^{\lambda\nu}$ through a unique ME-connection subject to the conditions of Agreement (4.1) is called *g-ME-manifold and we denote it by *g-MEX$_n$. The purpose of the present paper is to introduce this new concept of *g-MEX$_n$ and investigate its properties. In this paper, we first prove a necessary and sufficient condition for the unique existence of ME-connection in X_n, and derive a surveyable tensorial representation of the ME-connection. In the second, we investigate the conformal change of *g-MEX$_n$ and present a useful tensorial representation of the conformal change of the ME-connection.

KEY WORDS AND PHRASES. *g-MEX$_n$, ME-connection, ME-vector,conformal change.
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1. INTRODUCTION.

In Appendix II to his last book Einstein([9], 1950) proposed a new unified field theory that would include both gravitation and electromagnetism. Although the intent of this theory is physical, its exposition is mainly geometrical. It may be characterized as a set of geometrical postulates for the space time X_4. The geometrical consequences of these postulates are not developed very far by Einstein. Characterizing Einstein's unified field theory as a set of geometrical postulates in X_4, Hlavatý ([12], 1957) gave its mathematical foundation for the first time. Since then the geometrical consequences of these postulates are developed very far by number of mathematicians.

Generalizing X_4 to n-dimensional generalized Riemannian manifold X_n, n-dimensional generalization of this theory,so called "Einstein's n-dimensional unified field theory"(denoted by n-g-UFT in what follows), had been attempted by Wrede ([18], 1958) and Mishra ([16], 1959). Corresponding to n-g-UFT, Chung ([1], 1963) particularly introduced a new unified field theory, called "n-dimensional *g-unified field theory"(denoted by n-*g-UFT in what follows), which is more useful than n-g-UFT in some aspects. They published several papers([1]$\sim$ [4], 1963 $\sim$ 1981) concerning this theory, particularly proving that n-*g-UFT is equivalent to n-g-UFT so far as classes and indices of inertia are concerned.

However, neither of these two n-dimensional generalizations is capable of representing a general n-dimensional Einstein's connection in a surveyable tensorial form probably due to the complexities in higher demensions.

Recently, in order to solve the above difficulty Chung and et al ([5], 1987) introduced a new concept of n-dimensional SE-manifold in n-g-UFT, imposing the semi-symmetric condition on X_n , and found a unique representation of n-dimensional Einstein's connection in a beautiful and surveyable form. Since then, many results concerning this manifold have been obtained ([5] $\sim$ [8], 1987$\sim$1989).

On the other hand, Friedmann([10], 1924) and Schouten([17], 1954) also introduced the idea of semi-symmetric connection, and Hayden ([11], 1932) the concept of metric connection. Recently, Yano ([20], 1970), Yano & Imai ([19], 1975) and Imai ([13], 1972; [14], 1973) assigned a semi-symmetric metric connection to an n-dimensional Riemannian manifold and found many results concerning this manifold.

Recently, Ko ([15], 1987) and Yoo ([21]. 1988) introduced a new concept of ME-manifold in n-g-UFT, assigning to X_n a ME- connection which is similar to Yano and Imai's semi-symmetric metric connection, and investigated its curvature tensors and conformal change in n-g-UFT . The purpose to introduce this manifold is similar to Chung's purpose to introduce SE-manifold.

The purpose of the present paper is to introduce a new concept of the n-dimensional *g-ME-manifold (denoted by *g-MEX$_n$), assigning an Einstein's connection of the form (3.1) to X_n, called a ME-connection in what follows, and investigate its properties. This paper consists of five sections. The second section introduces some preliminary notations, definitions, and results. The third section concerns with a necessary and sufficient condition for the existence of unique ME-connection in n-*g-UFT.

The fourth section deals with a precise tensorial representation of the ME-connection in terms of ${}^*g^{\lambda\nu}$. In the last section, we investigate the conformal change of *g-MEX$_n$, with particular emphasis on the conformal invariants of *g-MEX$_n$. In this section, we display a surveyable tensorial representation of the conformal change of the ME-connection

2. PRELIMINARIES.

This section is a brief collection of the basic concepts, notations, and results, which are needed in our further considerations in the present paper. It is based on the results and symbolisms of Hlavatý([43],1957) and Chung([10], 1963; [13], 1981; [16], 1985).

A. n-DIMENSIONAL *g-UNIFIED FIELD THEORY.

Hlavatý characterized Einstein's 4-dimensional unified field theory (4-g-UFT) as a set of geometrical postulates in a space-time X_4 for the first time and gave its mathematical foundation. Generalizing this theory we may consider Einstein's n-dimensional unified field theory(n-g-UFT). Similarly, our n-dimensional *g-unified field theory(n-*g-UFT), initiated by Chung(1963) and originally suggested by Hlavatý(1957), is based on the following three principles.

Principle A. Let X_n be a n-dimensional generalized Riemannian manifold referred to a real coordinate system x^ν, which obeys coordinate transformation $x^\nu \to x^{\nu'}$ for which

$$Det(\frac{\partial x'}{\partial x}) \neq 0 \tag{2.1}$$

Let $g_{\lambda\mu}$ be a general real nonsymmetric tensor which may be decomposed into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$(*):

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \tag{2.2}$$

where

$$\mathfrak{g} = Det(g_{\lambda\mu}) \neq 0, \qquad \mathfrak{h} = Det(h_{\lambda\mu}) \neq 0 \tag{2.3}$$

The algebraic structure on X_n is imposed by the basic real tensor ${}^*g^{\lambda\nu}$, uniquely defined by

$$g_{\lambda\mu}{}^*g^{\lambda\nu} = g_{\mu\lambda}{}^*g^{\nu\lambda} = \delta_\mu{}^\nu \tag{2.4}$$

It may also be decomposed into its symmetric part ${}^*h^{\lambda\nu}$ and skew-symmetric part ${}^*k^{\lambda\nu}$:

$${}^*g^{\lambda\nu} = {}^*h^{\lambda\nu} + {}^*k^{\lambda\nu} \tag{2.5}$$

(*) Throughout the present paper,Greek indices are used for the holonomic components of tensors in X_n. They take the values 1,2,...,n, n>1, and follow the summation convention.

Since Det$({}^*h^{\lambda\nu}) \neq 0$, we may define a unique tensor ${}^*h_{\lambda\mu}$ by

$$ {}^*h_{\lambda\mu}{}^*h^{\lambda\nu} = \delta_\mu{}^\nu \tag{2.6}$$

In n-*g-UFT we use both ${}^*h^{\lambda\nu}$ and ${}^*h_{\lambda\mu}$ as the tensors for raising and/or lowering indices of all tensors defined in X_n in the usual manner. We then have

$$ {}^*k_{\lambda\mu} = {}^*k^{\rho\sigma}{}^*h_{\lambda\rho}{}^*h_{\mu\sigma}, \quad {}^*g_{\lambda\mu} = {}^*g^{\rho\sigma}{}^*h_{\lambda\rho}{}^*h_{\mu\sigma} \tag{2.7a}$$

so that

$$ {}^*g_{\lambda\mu} = {}^*h_{\lambda\mu} + {}^*k_{\lambda\mu} \tag{2.7b}$$

Principle B. The differential geometrical structure on X_n is imposed by the tensor ${}^*g^{\lambda\nu}$ by means of a connection $\Gamma_\lambda{}^\nu{}_\mu$ defined by a system of equations, so called Einstein's equations

$$ D_\omega{}^*g^{\lambda\mu} = -2S_{\omega\alpha}{}^{\mu}{}^*g^{\lambda\alpha}(*) \tag{2.8}$$

Here D_ω denotes the symbol of the covariant derivative with respect to $\Gamma_\lambda{}^\nu{}_\mu$ and $S_{\lambda\mu}{}^\nu$ is the torsion tensor of $\Gamma_\lambda{}^\nu{}_\mu$. The connection $\Gamma_\lambda{}^\nu{}_\mu$ satisfying (2.8) is called an Einstein's connection.

Principle C. In order to obtain ${}^*g^{\lambda\nu}$ involved in the solution for $\Gamma_\lambda{}^\nu{}_\mu$ centain conditions are imposed. These conditions may be condensed to

$$ S_\lambda \stackrel{\text{def}}{=} S_{\lambda\alpha}{}^\alpha = 0, \qquad R_{[\mu\lambda]} = \partial_{[\mu}Y_{\lambda]}, \qquad R_{(\mu\lambda)} = 0 \tag{2.9}$$

where Y_λ is an arbitrary vector and

$$ R_{\omega\mu\lambda}{}^\nu \stackrel{\text{def}}{=} 2(\partial_{[\mu}\Gamma_{|\lambda|}{}^\nu{}_{\omega]} + \Gamma_\alpha{}^\nu{}_{[\mu}\Gamma_{|\lambda|}{}^\alpha{}_{\omega]}) \tag{2.10}$$

$$ R_{\mu\lambda} \stackrel{\text{def}}{=} R_{\alpha\mu\lambda}{}^\alpha, \qquad V_{\omega\mu} \stackrel{\text{def}}{=} R_{\omega\mu\alpha}{}^\alpha \tag{2.11}$$

are curvature tensors of X_n.

In the following Remark, we state the main differences between n-g-UFT and n-*g-UFT.

REMARK 2.1. In $\begin{cases}\text{n-g-UFT}\\ \text{n-*g-UFT}\end{cases}$, the algebraic structure is imposed on X_n by the tensor $\begin{cases} g_{\lambda\mu} \\ {}^*g^{\lambda\nu}\end{cases}$, and $\begin{cases}\text{the tensor } h^{\lambda\nu} \text{ and } h_{\lambda\mu} \\ \text{the tensor } {}^*h^{\lambda\nu} \text{ and } {}^*h_{\lambda\mu}\end{cases}$ are used for raising and/or lowering the indices of tensors in X_n. On the other hand, in $\begin{cases}\text{n-g-UFT}\\ \text{n-*g-UFT}\end{cases}$ the differential geometrical structure is imposed

(*) It has been shown by Hlavatý(1957) that the system (2.8) is equivalent to

$$ D_\omega g_{\lambda\mu} = 2S_{\omega\mu}{}^\alpha g_{\lambda\alpha} \tag{2.8$'$}$$

or

$$ \partial_\omega g_{\lambda\mu} - \Gamma_\lambda{}^\alpha{}_\omega g_{\alpha\mu} - \Gamma_\omega{}^\alpha{}_\mu g_{\lambda\alpha} = 0 \tag{2.8$''$}$$

which is the original Einstein's equation.

on X_n by $\begin{cases} g_{\lambda\mu} \\ {}^*g^{\lambda\nu} \end{cases}$ through $\Gamma_\lambda{}^\nu{}_\mu$ satisfying $\begin{cases} (2.8)' \\ (2.8) \end{cases}$ Hence, if $\begin{cases} (2.8)' \\ (2.8) \end{cases}$ admits a solution $\Gamma_\lambda{}^\nu{}_\mu$, it will be expressed in terms of $\begin{cases} g^{\lambda\nu} \text{ in n-g-UFT} \\ {}^*g^{\lambda\nu} \text{ in n-}^*\text{g-UFT} \end{cases}$

B. SOME NOTATIONS AND RESULTS.

The following quantities are frequently used in our further consideration:

$$^*\mathfrak{g} = \mathrm{Det}(^*g_{\lambda\mu}), \quad ^*\mathfrak{h} = \mathrm{Det}(^*h_{\lambda\mu}), \quad ^*\mathfrak{k} = \mathrm{Det}(^*k_{\lambda\mu}) \tag{2.12a}$$

$$^*g = \frac{^*\mathfrak{g}}{^*\mathfrak{h}}, \qquad ^*k = \frac{^*\mathfrak{k}}{^*\mathfrak{h}} \tag{2.12b}$$

$$\sigma = \begin{cases} 0, & \text{if n is even} \\ 1, & \text{if n is odd} \end{cases} \tag{2.12c}$$

$$^{(0)*}k_\lambda{}^\nu = \delta_\lambda{}^\nu \quad ,^{(1)*}k_\lambda{}^\nu = {}^*k_\lambda{}^\nu, \quad ^{(p)*}k_\lambda{}^\nu = {}^{(p)*}k_\lambda{}^{\alpha\,(p-1)*}k_\alpha{}^\nu, \quad p = 1,2,\cdots \tag{2.12d}$$

$$K_p = {}^*k_{[\alpha_1}{}^{\alpha_1}\,{}^*k_{\alpha_2}{}^{\alpha_2}\cdots{}^*k_{\alpha_p]}{}^{\alpha_p} \quad ,p = 1,2,\cdots \tag{2.12c}$$

An eingenvector V^ν of $k_{\lambda\mu}$ which satisfies

$$(M^*h_{\lambda\nu} + {}^*k_{\lambda\nu})V^\nu = 0, \qquad \text{M is a scalar} \tag{2.13}$$

is called a basic vector of X_n, and the corresponding eigenvalue M of $k_{\lambda\mu}$ a basic scalar of X_n.

The following theorems have been proved already in a X_n ([10], 1963; [13], 1981).

THEOREM 2.1. The following relations hold in a X_n:

$$\begin{cases} K_0 = 1, K_n = {}^*k, & \text{if n is even} \\ K_p = 0, & \text{if p is odd} \end{cases} \tag{2.14a}$$

$$^*g = \sum_{s=0}^{n-\sigma} K_s \tag{2.14b}$$

$$\sum_{s=0}^{n-\sigma} K_s\,{}^{(n-s)*}k_\lambda{}^\nu = 0 \tag{2.14c}$$

Here and in what follows, the index s is assumed to take the values 0,2,4,$\cdots$ in the specified range.

THEOREM 2.2. The basic scalars M satisfy the following equation:

$$\sum_{s=0}^{n-\sigma} K_s M^{n-s} = 0 \tag{2.15}$$

THEOREM 2.3. If the system (2.8) admits a solution $\Gamma_\lambda{}^\nu{}_\mu$, it must be of the form

$$\Gamma_\lambda{}^\nu{}_\mu = {}^*\{\lambda{}^\nu{}_\mu\} + S_{\lambda\mu}{}^\nu + {}^*U^\nu{}_{\lambda\mu} \tag{2.16}$$

where $^*\{\lambda{}^\nu{}_\mu\}$ are the Christoffel symbols defined by $^*h_{\lambda\mu}$ and

$$^*U^\nu{}_{\lambda\mu} = S_{\beta(\lambda}{}^{\nu*}k_{\mu)}{}^\beta + S^\nu{}_{\beta(\lambda}{}^*k_{\mu)}{}^\beta - S^\beta{}_{(\lambda\mu)}{}^*k_\beta{}^\nu \tag{2.17}$$

3. THE ME-CONNECTION IN n-*g-UFT

In the section, we introduce the concept of ME-connection in n-*g-UFT and devote mainly to the proof of a necessary and sufficient condition for the existence of ME-connection in a general X_n.

DEFINITION 3.1. An Einstein's connection $\Gamma_\lambda{}^\nu{}_\mu$ of the form

$$\Gamma_\lambda{}^\nu{}_\mu = {}^*\{{}_\lambda{}^\nu{}_\mu\} + 2\delta_\lambda{}^\nu X_\mu - 2^*g_{\lambda\mu}X^\nu \tag{3.1}$$

for a non-null vector X_λ is called a ME-connection in n-*g-UFT, and X_λ the corresponding ME-vector.

If X_n admits a ME-connection $\Gamma_\lambda{}^\nu{}_\mu$, it must also be of the form (2.16). Hence , comparing (2.16) and (3.1) we have

$$S_{\lambda\mu}{}^\nu = 2\delta_{[\lambda}{}^\nu X_{\mu]} - 2^*k_{\lambda\mu}X^\nu \tag{3.2}$$

$${}^*U^\nu{}_{\lambda\mu} = 2\delta_{(\lambda}{}^\nu X_{\mu)} - 2^*h_{\lambda\mu}X^\nu \tag{3.3}$$

THEOREM 3.2. If X_n admits a ME-connection $\Gamma_\lambda{}^\nu{}_\mu$, the ME-vector X_λ satisfies the relation

$$2\delta_{(\lambda}{}^\nu X_{\mu)} - 2^*h_{\lambda\mu}X^\nu + 2\,{}^{(2)*}k_{\lambda\mu}X^\nu - {}^*h_{\lambda\mu}{}^*k_\alpha{}^\nu X^\alpha = 0 \tag{3.4)a}$$

or equivalently

$$2^*h_{\nu(\lambda}X_{\mu)} - 2^*h_{\lambda\mu}X_\nu + 2\,{}^{(2)*}k_{\lambda\mu}X_\nu + {}^*h_{\lambda\mu}{}^*k_\nu{}^\alpha X_\alpha = 0 \tag{3.4)b}$$

Proof. The relation (3.2) gives

$$S^\nu{}_{\lambda\mu} = \delta_\mu{}^\nu X_\lambda + 2^*k_\lambda{}^\nu X_\mu - {}^*h_{\lambda\mu}X^\nu \tag{3.5}$$

Substituting (3.2) and (3.5) into the righthand side of (2.17),we have

$${}^*U^\nu{}_{\lambda\mu} = -2X^{\nu\,(2)*}k_{\lambda\mu} + {}^*h_{\lambda\mu}{}^*k_\alpha{}^\nu X^\alpha \tag{3.6}$$

Our relation (3.4)a immediately follows by comparing (3.3) and (3.6). The equivalence of (3.4)a and (3.4)b is obvious.

THEOREM 3.3. A necessary and sufficient condition for the system (2.8) to admit a solution $\Gamma_\lambda{}^\nu{}_\mu$ of the form (3.1) is

$$\nabla_\omega{}^*k^{\lambda\mu} = -2^*g^{\lambda\mu}X_\omega + 2\delta_\omega{}^\lambda X^\mu + 2\delta_\omega{}^{\mu*}k_\alpha{}^\lambda X^\alpha - 4\,{}^{(2)*}k_\omega{}^{(\lambda}X^{\mu)} \tag{3.7)a}$$

or equivalently

$$\nabla_\omega{}^*k_{\lambda\mu} = 2({}^*h_{\omega[\lambda}X_{\mu]} - {}^*k_{\lambda\mu}X_\omega + {}^*h_{\omega[\lambda}{}^*k_{\mu]}{}^\alpha X_\alpha) \tag{3.7)b}$$

Proof. Suppose that the system (2.8) admits a solution $\Gamma_\lambda{}^\nu{}_\mu$ of the form (3.1). Substituting (3.1) into the left-hand side of (2.8), we have

$$\begin{aligned} D_\omega{}^*g^{\lambda\mu} &= \partial_\omega{}^*g^{\lambda\mu} + \Gamma_\alpha{}^\lambda{}_\omega{}^*g^{\alpha\mu} + \Gamma_\alpha{}^\mu{}_\omega{}^*g^{\lambda\alpha} \\ &= \nabla_\omega{}^*k^{\lambda\mu} + 4^*g^{\lambda\mu}X_\omega - 4\delta_\omega{}^{(\lambda}X^{\mu]} + 4^*k_\omega{}^\lambda X^\mu - 4\,{}^{(2)*}k_\omega{}^{[\lambda}X^{\mu]} \end{aligned} \tag{3.8)a}$$

On the other hand, substitution of (3.2) into the right-hand side of (2.8) gives

$$-2S_{\omega\alpha}{}^{\mu}{}^*g^{\lambda\alpha} = -2\delta_\omega{}^\mu X^\lambda + 2^*g^{\lambda\mu}X_\omega + 4^*k_\omega{}^\lambda X^\mu - 2\delta_\omega{}^{\mu*}k^{\lambda\alpha}X_\alpha - 4\,{}^{(2)*}k_\omega{}^\lambda X^\mu \tag{3.8)b}$$

The condition (3.7)a results from (3.8)a and (3.8)b. Conversely, assume now that the condition (3.7)a is satisfied. Define a connection $\Gamma_\lambda{}^\nu{}_\mu$ by (3.1), and substitute it into both sides of (2.8). This substitution yields the relation (3.7)a, which is satisfied by our assumption. Hence, a connection of the form (3.1) is an Einstein's connection under the condition (3.7)a. In order to show the equivalence of (3.7)a and (3.7)b, consider the following alternative form (3.7)a:

$$\nabla_\omega {}^*k_{\lambda\mu} = -2{}^*g_{\lambda\mu}X_\omega + 2{}^*h_{\omega\lambda}X_\mu - 2{}^*h_{\omega\mu}{}^*k_\lambda^\alpha X_\alpha - 4\,{}^{(2)*}k_{\omega(\lambda}X_{\mu)} \tag{3.9}$$

The equivalence immediately follows from (3.9) in virtue of (3.4)b.

REMARK 3.4. Using (3.4)b and (3.9), it can be easily shown that the following relation is also equivalent to (3.7)a:

$$\nabla_\omega {}^*k_{\lambda\mu} = 2(4{}^*k_{\omega[\lambda}X_{\mu]} - {}^*k_{\lambda\mu}X_\omega - 2\,{}^{(2)*}k_{\omega[\lambda}X_{\mu]}) \tag{3.10}$$

In our further considerations we shall assume that the tensor

$$A_{\lambda\mu} \overset{\text{def}}{=} -n{}^*g_{\lambda\mu} + {}^*g_{\mu\lambda} \tag{3.11}$$

is of rank n, so that there exists a unique tensor $B^{\lambda\nu}$ satisfying the condition

$$A_{\lambda\mu}B^{\lambda\nu} = A_{\mu\lambda}B^{\nu\lambda} = \delta_\mu{}^\nu \tag{3.12}$$

THEOREM 3.5. A necessary and sufficient condition for the system (2.8) to admit exactly one ME-connection $\Gamma_\lambda{}^\nu{}_\mu$ of the form (3.1) is that the tensor field ${}^*g^{\lambda\nu}$ satisfies the following condition:

$$\nabla_\omega {}^*k_{\lambda\mu} = 2({}^*h_{\omega[\lambda}{}^*g_{\mu]\beta} - {}^*h_{\omega\beta}{}^*k_{\lambda\mu})C_\alpha B^{\alpha\beta} \tag{3.13}$$

If this condition is satisfied, then

$$X^\nu = C_\alpha B^{\alpha\nu} \tag{3.14}$$

where

$$C_\lambda = \nabla_\alpha {}^*k_\lambda{}^\alpha \tag{3.15}$$

Proof. If the system (2.8) admits a solution $\Gamma_\lambda{}^\nu{}_\mu$ of the form (3.1), the condition (3.7)b must hold in virtue of Theorem (3.3). Raising the index μ and putting $\omega = \mu = \alpha$ in (3.7)b we have

$$C_\lambda = A_{\lambda\alpha}X^\alpha \tag{3.16}$$

in virtue of (3.11) and (3.15). Multiplication of $B^{\lambda\nu}$ to both sides of (3.16) gives (3.14) in virtue of (3.12). The condition (3.13) now follows by substituting (3.14) into (3.7)b. The proof of the converse statement is obvious in virtue of Theorem (3.3).

Besides the ME-connection $\Gamma_\lambda{}^\nu{}_\mu$ given by (3.1) and (3.14), assume that there exists another ME-connection

$$\widehat{\Gamma}_\lambda{}^\nu{}_\mu = {}^*\{\lambda{}^\nu{}_\mu\} + 2\delta_\lambda{}^\nu \widehat{X}_\mu - 2{}^*g_{\lambda\mu}\widehat{X}_\nu, \qquad \widehat{X}_\lambda \neq X_\lambda \tag{3.17}$$

Applying the same method to derive (3.14) to this connection, we have

$$\widehat{X}_\lambda = C_\alpha B^{\alpha\nu} = X_\lambda$$

which is a contradiction to our assumption (3.17). This proves the uniqueness of the ME-connection under the condition (3.13).

4. *g-ME-MANIFOLD AND A REPRESENTATION OF ITS CONNECTION.

In this section, we introduce the concept of *g-ME-manifold and display a surveyable tensorial representation of a unique ME-connection $\Gamma_\lambda{}^\nu{}_\mu$ in terms of the tensor field ${}^*g^{\lambda\nu}$ using two useful recurrence relations.

AGREEMENT4.1. In our further considerations in the present paper, we impose priori the following conditions to the tensor field ${}^*g^{\lambda\nu}$:

(a) The quantity

$$\theta \stackrel{\text{def}}{=} \frac{1-n}{1+n}(\neq 0) \tag{4.1}$$

is not a basic scalar of X_n(see (2.13)).

(b) The condition (3.13) is satisfied by the tensor field ${}^*g^{\lambda\nu}$

According to the condition (b), we note that there always exists a unique ME-connection $\Gamma_\lambda{}^\nu{}_\mu$ in our n-*g-UFT. In virtue of (3.1) and (3.14), this connection may be given by

$$\Gamma_\lambda{}^\nu{}_\mu = {}^*\{\lambda{}^\nu{}_\mu\} + 2(\delta_\lambda{}^{\nu*}h_{\mu\beta} - {}^*g_{\lambda\mu}\delta_\beta{}^\nu)C_\alpha B^{\alpha\beta} \tag{4.2}$$

DEFINITION 4.2. An n-dimensional generalized Riemannian manifold X_n, on which the differential geometric structure is imposed by the tensor ${}^*g^{\lambda\nu}$ satisfying the conditions of Agreement (4.1) by means of the unique ME-connection given by (4.2), is called an n-dimensional *g-ME-manifold and denoted by *g-MEX$_n$.

In our further considerations in the present paper, we use the following useful abbreviations for an arbitrary vector V_λ, where p=2,3,4,$\cdots$:

$$^{(1)}V_\lambda \stackrel{\text{def}}{=} V_\lambda, \quad ^{(p)}V_\lambda \stackrel{\text{def}}{=} {}^{(p-1)*}k_\lambda{}^\alpha\ V_\alpha \tag{4.3)a}$$

In virtue of (2.12)d, we then have

$$^{(p)}V_\lambda = {}^*k_\lambda{}^\alpha\ {}^{(p-1)}V_\alpha \tag{4.3)b}$$

$$^{(1)}V^\nu = V^\nu, \quad ^{(p)}V^\nu = {}^{(p-1)*}k^\nu{}_\alpha\ V^\alpha = {}^*k^\nu{}_\alpha\ {}^{(p-1)}V^\alpha \tag{4.3)c}$$

For s=2,4,$\cdots$,n-σ, we also use the following quantities:

$$H_0 = k_0 = 1, \qquad H_s = \theta^2\ H_{s-2} + K_s \tag{4.4}$$

In virtue of (2.12)c and (2.14)a, direct calculations show that

$$H_s = \sum_{t=0}^{s} K_t\ \theta^{s-t} \tag{4.5}$$

In particular,

$$H_{n-\sigma} = \sum_{s=0}^{n-\sigma} K_s\ \theta^{n-s-\sigma} \tag{4.6}$$

THEOREM 4.3. In *g-MEX$_n$ the following recurrence relation holds:

$$^{(p)}X_\lambda = \theta\,^{(p-1)}X_\lambda - \frac{1}{1+n}\,^{(p-1)}C_\lambda (p=2,\ 3,\ 4,\cdots) \tag{4.7}$$

where θ and C_λ are respectively given by (4.1) and (3.15).

Proof. Substituting (3.11) into (3.16), we have

$$ {}^{(2)}X_\lambda = \theta X_\lambda - \frac{1}{1+n} C_\lambda \tag{4.8} $$

The relation (4.7) now follows by multiplying ${}^{(p-2)*}k_\mu{}^\lambda$ to both sides of (4.8) and making use of the abbreviations (4.3).

THEOREM 4.4. In *g-MEX$_n$ the following recurrence relation holds:

$$ \sum_{s=0}^{n-\sigma} K_s \, {}^{(n-s+1)}X_\lambda = 0 \tag{4.9} $$

Proof. This relation follows by multiplying X_ν to both sides of (2.14)c and making use of the abbreviations (4.3).

Application of the recurrence relations (4.7) and (4.9) yields the following surveyable representation of the ME-vector X_λ.

THEOREM 4.5. In *g-MEX$_n$ the ME-vector X_λ may be given by

$$ X_\lambda = \frac{1}{\phi}\left(\sigma H_{n-\sigma} C_\lambda + \sum_{s=0}^{n-2-\sigma} H_s \, {}^{(n-s)}Q_\lambda\right) \tag{4.10} $$

where

$$ \phi \stackrel{\text{def}}{=} (1+n)\theta^\sigma H_{n-\sigma}, \tag{4.11} $$

$$ {}^{(p)}Q_\lambda \stackrel{\text{def}}{=} {}^{(p)}C_\lambda + \theta \, {}^{(p-1)}C_\lambda, \qquad p = 2, 3, \cdots, n \tag{4.12} $$

Proof. Substitute ${}^{(n+1)}X_\lambda$ from (4.7) into the first term of (4.9) to obtain

$$ \frac{1}{1+n} {}^{(n)}C_\lambda = \theta \, {}^{(n)}X_\lambda + \sum_{s=2}^{n-\sigma} K_s \, {}^{(n-s+1)}X_\lambda \tag{4.13a} $$

Substituting ${}^{(n)}X_\lambda$ again from (4.7) into the first term on the right-hand side of (4.13)a and making use of (4.4) and (4.12), we have

$$ \frac{1}{1+n} {}^{(n)}Q_\lambda = H_2 \, {}^{(n-1)}X_\lambda + \sum_{s=4}^{n-\sigma} K_s \, {}^{(n-s+1)}X_\lambda \tag{4.13b} $$

Similarly, the second step may be carried out as in the above first step. The substitution of ${}^{(n-1)}X_\lambda$ into (4.13)b in the first and that of ${}^{(n-2)}X_\lambda$ into the resulting relation in the second give

$$ \frac{1}{1+n}\left({}^{(n)}Q_\lambda + H_2 {}^{(n-2)}Q_\lambda\right) = H_4 \, {}^{(n-3)}X_\lambda + \sum_{s=6}^{n-\sigma} K_s \, {}^{(n-s+1)}X_\lambda \tag{4.14} $$

After $\frac{n-\sigma}{2}$ steps of successive repeated substitution for ${}^{(p)}X$ from (4.7), we have

$$ \frac{1}{1+n}\left({}^{(n)}Q_\lambda + H_2 \, {}^{(n-2)}Q_\lambda + \cdots + H_{n-2-\sigma} \, {}^{(2+\sigma)}Q_\lambda\right) = H_{n-\sigma} \, {}^{(1+\sigma)}X_\lambda \tag{4.15} $$

On the other hand, it follows from (2.12)c and (4.8) that

$$^{(1+\sigma)}X_\lambda = \theta^\sigma X_\lambda - \frac{\sigma}{1+n} C_\lambda \tag{4.16}$$

Our representation (4.10) immediately follows by substituting (4.16) into (4.15) and making use of (4.11).

REMARK 4.6. According to the Agreement (4.1)a, θ is not a basic scalar of X_n.

Hence, in *g-MEX$_n$ the following relation always holds in virtue of (2.15) and (4.6):

$$H_{n-\sigma} \neq 0 \tag{4.17}$$

We note that the relation (4.17) justifies the validity of the representation (4.10). Furthermore, we also note that the condition (4.17) is a necessary and sufficient condition for X_n to admit a unique ME-vector X_λ in n-*g-UFT. This is the reason why we imposed prior the condition of Agreement (4.1)(a).

Now that we have obtained a representation of the ME-vector X_λ in terms of $^*g^{\lambda\nu}$, it is possible for us to obtain a surveyable representation of the ME-connection of *g-MEX$_n$ in terms of $^*g^{\lambda\nu}$ by simply substituting (4.10) into (3.1). Formally we state

THEOREM 4.7. The ME-connection of *g-MEX$_n$ may be given by

$$\Gamma_\lambda{}^\nu{}_\mu = {}^*\{\lambda{}^\nu{}_\mu\} + \frac{2}{\phi} G^{\nu\alpha}_{\lambda\mu}\left(\sigma H_{n-\sigma}\, C_\alpha + \sum_{s=0}^{n-2-\sigma} H_s\, {}^{(n-s)}Q_\alpha\right) \tag{4.18}$$

where the vectors C_λ and $^{(p)}Q_\alpha$ are respectively defined by (3.15) and (4.12), the quantities $H_{n-\sigma}$ and ϕ by (4.4) and (4.11) respectively, and

$$G^{\nu\alpha}_{\lambda\mu} \stackrel{\text{def}}{=} \delta_\lambda{}^\nu \delta_\mu{}^\alpha - {}^*g_{\lambda\mu}{}^*h^{\nu\alpha} \tag{4.19}$$

5. CONFORMAL CHANGE OF *g-MEX$_n$.

In this section, we investigate change of several geometrical quantities, particular emphasis on the change of the unique ME-vector and ME-connection, induced by a conformal change of the unified field tensor $^*g^{\lambda\nu}$

Let $\left\{\begin{array}{l} {}^*g - MEX_n \\ \overline{{}^*g - MEX_n} \end{array}\right.$ be n-dimensional *g-ME-manifold, on which differential geometric structure is imposed by a unified field tensor field $\left\{\begin{array}{l} {}^*g^{\lambda\nu} \\ {}^*\bar{g}^{\lambda\nu} \end{array}\right.$ through the ME-connection $\left\{\begin{array}{l} \Gamma_\lambda{}^\nu{}_\mu \text{ given by (4.18)} \\ \overline{\Gamma}_\lambda{}^\nu{}_\mu \text{ given by (5.1)} \end{array}\right.$:

$$\overline{\Gamma}_\lambda{}^\nu{}_\mu = {}^*\overline{\{\lambda{}^\nu{}_\mu\}} + \frac{2}{\bar\phi} \overline{G}^{\nu\alpha}_{\lambda\mu}\left(\sigma \overline{H}_{n-\sigma}\overline{C}_\alpha + \sum_{s=0}^{n-2-\sigma} \overline{H}_s\, {}^{(n-s)}\overline{Q}_\alpha\right) \tag{5.1}$$

(See Agreement (5.2) for $\bar\phi, \overline{G}^{\nu\alpha}_{\lambda\mu}, \overline{H}_s, \overline{C}_\alpha$, and $\overline{Q}_\alpha$)

DEFINITION 5.1. Two manifolds $^*g$-MEX_n and $\overline{{}^*g - MEX_n}$ are said to be conformal, if their basic tensor fields are related by

$${}^*\bar{g}^{\lambda\nu}(x) = e^{-\Omega}\, {}^*g^{\lambda\nu}(x) \, (*) \tag{5.2}$$

(*) Note that the conformal change defined by (5.2) is equivalent to $\bar{g}_{\lambda\mu}(x) = e^{\Omega} g_{\lambda\mu}(x)$, which is the definition of conformal change in n-g-UFT where $\Omega = \Omega(x)$ is an arbitrary function of position with at least two derivatives.

This conformal change enforces a change of ME-vector and ME-connection, and an explicit tensorial representation of this change will be displayed in this section.

AGREEMENT 5.2. Throughout this section, we agree that, if T is a function of ${}^*g_{\lambda\mu}$, then we denote by $\overline{T}$ the same function of ${}^*\overline{g}_{\lambda\mu}$. In particular, if T is a tensor, so is $\overline{T}$. Furthermore, the indices of T($\overline{T}$) will be raised and/or lowered by means of ${}^*h^{\lambda\nu}$ (${}^*\overline{h}^{\lambda\nu}$) and/or ${}^*h_{\lambda\mu}$ (${}^*\overline{h}_{\lambda\mu}$) .

The following two theorems are immediate consequences of Definition(5.1) and Agreement(5.2).

THEOREM 5.3. The conformal change (5.2) induces the following changes:

$$ {}^*\overline{h}^{\lambda\nu} = e^{-\Omega}\ {}^*h^{\lambda\nu}, \qquad {}^*\overline{h}_{\lambda\mu} = e^{\Omega}\ {}^*h_{\lambda\mu} \tag{5.3)a} $$

$$ {}^{(p)*}\overline{k}^{\lambda\nu} = e^{-\Omega}\,{}^{(p)*}k^{\lambda\nu}, \qquad {}^{(p)*}\overline{k}_{\lambda\mu} = e^{\Omega}\,{}^{(p)*}k_{\lambda\mu}\ , \quad (p = 1, 2, \cdots) \tag{5.3)b} $$

$$ {}^*\overline{\mathfrak{g}} = e^{n\Omega}\,{}^*\mathfrak{g}, \quad {}^*\overline{\mathfrak{h}} = e^{n\Omega}\,{}^*\mathfrak{h}, \quad {}^*\overline{\mathfrak{k}} = e^{n\Omega}\,{}^*\mathfrak{k} \tag{5.3)c} $$

THEOREM 5.4. The tensors ${}^{(p)*}k_\lambda{}^\nu$ and $G^{\nu\alpha}_{\lambda\mu}$ and the quantities ${}^*g\, ,{}^*k\, ,K_s\, ,H_s\, ,\theta$ and ϕ are conformal invariants with respect to (5.2). That is,

$$ {}^{(p)*}\overline{k}_\lambda{}^\nu = {}^{(p)*}k_\lambda{}^\nu \quad (p = 0, 1, 2, \cdots) \tag{5.4)a} $$

$$ \overline{G}^{\nu\alpha}_{\lambda\mu} = G^{\nu\alpha}_{\lambda\mu} \tag{5.4)b} $$

$$ \overline{K}_s = K_s\ , \overline{H}_s = H_s\ , {}^*\overline{g} = {}^*g\ , {}^*\overline{k} = {}^*k\ , {}^*\overline{\theta} = \theta\ , {}^*\overline{\phi} = \phi \tag{5.4)c} $$

THEOREM 5.5. The conformal change (5.2) induces the following changes:

$$ {}^*\overline{\{\lambda{}^\nu{}_\mu\}} = {}^*\{\lambda{}^\nu{}_\mu\} + \delta_{(\lambda}{}^\nu\Omega_{\mu)} - \frac{1}{2}{}^*h_{\lambda\mu}\Omega^\nu \tag{5.5)a} $$

$$ \overline{\nabla}_\nu{}^*\overline{k}_{\lambda\mu} = e^{\Omega}\left(\nabla_\nu{}^*k_{\lambda\mu} + {}^*k_{\nu[\lambda}\Omega_{\mu]} - {}^*h_{\nu[\lambda}{}^{(2)}\Omega_{\mu]}\right) \tag{5.5)b} $$

$$ {}^{(p)}\overline{C}_\lambda = {}^{(p)}C_\lambda + \frac{n-2}{2}\,{}^{(p+1)}\Omega_\lambda, \quad (p = 1, 2, \cdots) \tag{5.5)c} $$

where

$$ \Omega_\lambda \stackrel{\text{def}}{=} \partial_\lambda\Omega \tag{5.6} $$

Proof. Substituting (5.3)a into

$$ {}^*\overline{\{\lambda{}^\nu{}_\mu\}} = \frac{1}{2}{}^*\overline{h}^{\nu\alpha}\left(\partial_\mu{}^*\overline{h}_{\lambda\alpha} + \partial_\lambda{}^*\overline{h}_{\mu\alpha} - \partial_\alpha{}^*\overline{h}_{\lambda\mu}\right) $$

the relation (5.5)a immediately follows in virtue of (5.6). Similarly, the change (5.5)b may be shown by substituting (5.3)b and (5.5)a into

$$ \overline{\nabla}_\nu{}^*\overline{k}_{\lambda\mu} = \partial_\nu{}^*\overline{k}_{\lambda\mu} - {}^*\overline{\{\lambda{}^\alpha{}_\nu\}}{}^*\overline{k}_{\alpha\mu} - {}^*\overline{\{\mu{}^\alpha{}_\nu\}}{}^*\overline{k}_{\lambda\alpha} $$

Making use of (3.15) and (5.3)a, the change (5.5)c for $p = 1$ may be obtained from (5.5)b as in the following way:

$$ \overline{C}_\lambda = {}^*\overline{h}^{\nu\alpha}\overline{\nabla}_\nu{}^*\overline{k}_{\lambda\alpha} = C_\lambda + \frac{n-2}{2}\,{}^{(2)}\Omega_\lambda \tag{5.7} $$

Our assertion (5.5)c immediately follows from (5.7) in virtue of (4.3)a.

Finally, in order to exhibit simplified representations of the conformal changes $X_\lambda \to \overline{X}_\lambda$ of the unique ME-vector and $\Gamma_\lambda{}^\nu{}_\mu \to \overline{\Gamma}_\lambda{}^\nu{}_\mu$ of the ME-connection, we use the following vector P_λ in our further considerations:

$$P_\lambda \overset{\text{def}}{=} \frac{n-2}{\phi}\left(\sigma H_{n-\sigma}{}^{(2)}\Omega_\lambda + \sum_{s=0}^{n-2-\sigma} H_s\left({}^{(n-s+1)}\Omega_\lambda + \theta^{(n-s)}\Omega_\lambda\right)\right) \tag{5.8}$$

THEOREM 5.9. The ME-vector X_λ and the ME- connection $\Gamma_\lambda{}^\nu{}_\mu$ are respectively transformed by the conformal change (5.2) as follows:

$$\overline{X}_\lambda = X_\lambda + \frac{1}{2}P_\lambda \tag{5.9}$$

$$\overline{\Gamma}_\lambda{}^\nu{}_\mu = \Gamma_\lambda{}^\nu{}_\mu + \delta_{(\lambda}{}^\nu\Omega_{\mu)} - \frac{1}{2}{}^*h_{\lambda\mu}\Omega^\nu + G^{\nu\alpha}_{\lambda\mu}P_\alpha \tag{5.10}$$

Proof. In virtue of (4.12) and (5.5)c, we first note that the change of the vector ${}^{(p)}Q_\lambda$ may be given by

$${}^{(p)}\overline{Q}_\lambda = {}^{(p)}Q_\lambda + \frac{n-2}{2}\left({}^{(p+1)}\Omega_\lambda + \theta^{(p)}\Omega_\lambda\right), \quad p = 2, 3, \cdots \tag{5.11}$$

After a length and tedious calculation, it may be easily proved that the relation (5.9) and (5.10) follows from (4.10) and (5.1) respectively making use of (5.4), (5.5),(5.8),and (5.11).

When $n = 2$, we have very interesting results as in the following theorem.

THEOREM 5.10. In a *g-MEX$_2$, the vectors ${}^{(p)}C_\lambda$ and ${}^{(p)}X_\lambda$ are conformal invariants with respect to the conformal change (5.2). That is,

$${}^{(p)}\overline{C}_\lambda = {}^{(p)}C_\lambda, \quad {}^{(p)}\overline{X}_\lambda = {}^{(p)}X_\lambda, \ (p = 1, 2, \cdots) \tag{5.12}$$

connection is given by

$$\overline{\Gamma}_\lambda{}^\nu{}_\mu = \Gamma_\lambda{}^\nu{}_\mu + \delta_{(\lambda}{}^\nu\Omega_{\mu)} - \frac{1}{2}{}^*h_{\lambda\mu}\Omega^\nu \tag{5.13}$$

Therefore, its torsion tensor is also a conformal invariant. That is,

$$\overline{S}_{\lambda\mu}{}^\nu = S_{\lambda\mu}{}^\nu \tag{5.14}$$

Proof. Since $P_\lambda = 0$ when $n = 2$, the relations (5.12) follows immediately from (5.5)c and (5.9). The relation (5.13) is a direct consequence of (5.10).

REMARK 5.11. Although the relations (5.12) hold, it should be noted that the vectors ${}^{(p)}C^\nu$ and ${}^{(p)}X^\nu$ are not conformal invariants in *g-MEX$_2$. In fact, the changes of these vectors are given by

$${}^{(p)}\overline{C}^\nu = e^{-\Omega}\,{}^{(p)}C^\nu, \quad {}^{(p)}\overline{X}^\nu = e^{-\Omega}\,{}^{(p)}X^\nu, \ (p = 1, 2, \cdots) \tag{5.15}$$

REMARK 5.12. In virtue of the second relations of (5.12) and (5.15), we also note that the tensor ${}^*U^\nu{}_{\lambda\mu}$ given by (3.3) is also conformal invariant in *g-MEX$_2$. That is,

$$\overline{U}^\nu{}_{\lambda\mu} = U^\nu{}_{\lambda\mu} \tag{5.16}$$

REMARK (5.13). An Einstein's connection $\Gamma_\lambda{}^\nu{}_\mu$ whose torsion tensor is of the form

$$S_{\lambda\mu}{}^\nu = 2\delta_{[\lambda}{}^\nu X_{\mu]}$$

is called a SE-connection in n-*g-UFT. In one of our unpublished papers, we have shown that this connection exists uniquely under a certain condition and that the conformal change of 2-dimensional SE-connection is given by

$$\overline{\Gamma}_\lambda{}^\nu{}_\mu = \Gamma_\lambda{}^\nu{}_\mu + \delta_{(\lambda}{}^\nu \Omega_{\mu)} - {}^*h_{\lambda\mu}\left(\frac{1}{2}\Omega^\nu - \frac{1-e^\Omega}{{}^*g}{}^{(2)}C^\nu\right) \tag{5.17}$$

in 2-*g-UFT. Note the difference between (5.13) and (5.17).

REFERENCES

1. CHUNG, K.T. Einstein's connection in terms of ${}^*g^{\lambda\nu}$, Nuovo Cimento, (X),27,(1963), 1279-1324
2. CHUNG, K.T. and CHANG, K.S. Degenerate cases of the Einstein's connection in the *g-unified field theory.-I, Tensor (New series), 20, No.2,(1969),143-149
3. CHUNG, K.T. and HAN, T.S. n-dimensional representations of the unified field tensor ${}^*g^{\lambda\nu}$, International Journal of Theoretical Physics, 20, No.10,(1981),739-747
4. CHUNG, K.T. and CHEOI, D.H. A study on the relations of two n-dimensional unified theories, Acta Mathematica Hungarica, 45 (1-2),(1985),141-149
5. CHUNG, K.T. and CHO, C.H. On the n-dimensional SE-connection and its conformal change, Nuovo Cimento, 100B, N0.4, (1987),537-550
6. CHUNG, K.T. and LEE, I.Y. Curvature tensors and unified field equations on SEX, International Journal of Theoretical Physics, 27, No.9, (1988), 1083-1104
7. CHUNG, K.T., SO, K.S. and LEE, J.W. Geometry of the submanifolds of SEX_n.-I. The C-nonholonomic frame of reference, International Journal of Theoretical Physics, 28, No.8, (1989),851-866
8. CHUNG, K.T. and LEE, J.W. Geometry of the submanifolds of SEX_n.-II. The generalized fundamental equations for the hypersubmanifold of SEX_n.International Journal of Theoretical Physics, 28, No.8, (1989),867-873
9. EINSTEIN, A. The meaning of relativity, Princeton University Press, 1950
10. FRIEDMANN, A. and SCHOUTEN, J.A. Über die geometrie der halbsymmetrischen Übertragung, Math. Zeitschr., 21, 1924
11. HAYDEN, H.A. Subspaces of a space with torsion, Proc. London Math. Soc.,34 1932
12. HLAVATÝ, V. Geometry of Einstein's unified field theory, Noordhoop Ltd., 1957
13. IMAI, T. Notes on semi-symmetric metric connection, Tensor (New Series), 24, 1972
14. IMAI, T. Notes on semi-symmetric metric connectionS II, Tensor (New Series), 27, 1973
15. KO, J.M. A Study on the curvature tensors in a manifold MEX_n, Ph.d. Thesis, Graduate School, Yonsei University, 1987
16. MISHRA, R.S. n-dimensional considerations of unified field theory of relativity, Tensor, 9,(1959), 217-225
17. SCHOUTEN, J.A. Ricci Calculus, Springer-Verlag, Berlin, 1954
18. WERDE, R.C. n-dimensional considerations of the basic principles A and B of the unified theory of relativity, Tensor,8, (1958), 95-122
19. YANO, K. and IMAI, T. On semi-symmetric metric F-connection, Tensor (New Series), 29, 1975
20. YANO, K. On semi-symmetric metric connections, Revue Roumaine de Mathématiques pures et Appliquées, 15, 1970
21. YOO, K.J. A study on the ME-connection and its conformal change, Ph.D. Thesis, Graduate School, Yonsei University, 1988