Internat. J. Math. & Math. Sci.
VOL. 17 NO. 2 (1994) 323-340

ON FELLER'S CRITERION FOR THE LAW OF THE ITERATED LOGARITHM

DELI LI

Institute of Mathematics
Jilin University
Changchun 130023
China

and

Department of Mathematics & Statistics
University of Regina
Regina, Saskatchewan
Canada, S4S 0A2

M. BHASKARA RAO

Department of Statistics
North Dakota State University
Fargo, ND 58105
USA

XIANGCHEN WANG

Department of Mathematics
Jilin University
Changchun 130023
China

(Received February 4, 1992 and in revised form February 12, 1993)

ABSTRACT. Combining Feller's criterion with a non-uniform estimate result in the context of the Central Limit Theorem for partial sums of independent random variables, we obtain several results on the Law of the Iterated Logarithm. Two of these results refine corresponding results of Wittmann (1985) and Egorov (1971). In addition, these results are compared with the corresponding results of Teicher (1974), Tomkins (1983) and Tomkins (1990)

KEY WORDS AND PHRASES. Central Limit Theorem, Feller's Criterion, Law of the Iterated Logarithm, Non-uniform estimates in the Central Limit Theorem.

1991 AMS SUBJECT CLASSIFICATION CODE. 60F15, 60F05.

1. INTRODUCTION

Let X_n, $n\geq 1$ be a sequence of independent random variables with zero means and finite variances. Let, for each $n \geq 1$,

$$S_n = \sum_{i=1}^n X_i, \quad s_n^2 = \sum_{i=1}^n EX_i^2, \quad F_n(x) = P(S_n/s_n \leq x),$$

$$\text{and} \quad \Phi(x) = \int_{-\infty}^{x} (1/\sqrt{2\pi}) \exp(-t^2/2)\, dt \quad \text{for} \quad -\infty < x < \infty.$$

Feller (1970) proved the following remarkable criterion on the Law of the Iterated Logarithm.

THEOREM A. Let X_n, $n\geq 1$ be a sequence of independent random variables with zero means and finite variances, and a_n, $n\geq 1$ a sequence of positive numbers such that $a_n/s_n \uparrow \infty$. If there exists a $\rho \geq 0$ such that

$$\sum_{n\geq 1} \min\{1, (a_{n+1} - a_n)/a_n\} P(S_n > a_n x) \begin{cases} < \infty & \text{for } x > \rho, \\ = \infty & \text{for } x < \rho, \end{cases}$$

then

$$\limsup_{n\to\infty} S_n/a_n = \rho \quad \text{a.s.}$$

Let G be the collection of all real valued functions g of a real variable with the following properties.

(a) g is nonnegative, even and nondecreasing in the interval $(0, \infty)$.

(b) $x/g(x)$ is nondecreasing in the interval $(0, \infty)$.

Combining Feller's criterion with a non-uniform estimate result in the context of the Central Limit Theorem for partial sums of independent random variables, we obtain several results on the asymptotic behavior of S_n/a_n with $a_n \uparrow \infty$ (Theorems 2.2, 2.3, 2.4 and 2.5). We would like to discuss some results of this genre available in the literature and compare them with the results we have derived in this paper. Theorem 1.1 of Wittmann (1985), a certain result of Egorov (1971), Theorem 1 of Teicher (1974), Theorem 1.1 of Tomkins (1983) and Theorem 1 of Tomkins (1990) are some of the results we seek out for comparison. Theorems 2.2 and 2.3 improve the general Law of the Iterated Logarithm due to Wittmann (1985) from which many classical results follow. Theorem 2.4 deals with the case of random variables X_n, $n \geq 1$

stochastically dominated by a nonnegative random variable X and improves the corresponding theorem due to Egorov (1971) by weakening the condition from $EX^2 < \infty$ to $E(X^2/L_2X) < \infty$. (Here, and in what follows, Lx stands for $\log \max(x,e)$ and L_nx stands for $L(L_{n-1}x)$ for $n\geq 2$.) In the same vein, two examples are given to compare Theorem 2.5 with Egorov's theorem. Example 4 shows that the conditions in Theorem 2.5 are, in some sense, optimal. Recently, Tomkins (1990) obtained a general result on the Law of the Iterated Logarithm. In Tomkins's paper, no assumptions have been made about the existence of any moments of X_n, $n\geq 1$. Our condition (2.3) is similar to Tomkins's condition (5) or (5)′ However, our conditions are more easily verifiable. Later, we provide an example satisfying the conditions of Theorem 2.2 and thus of Theorem 2.3 but not those of Theorem 1 of Teicher (1974) and those of Theorem 1.1 of Tomkins (1983). See Example 1 below. We will also provide another example which satisfies the conditions of Theorem 1 of Teicher (1974) as well as the conditions of Theorem 1.1 of Tomkins (1983) but not those of Theorem 2.2. Some additional comments will be made as and when the occasion arises.

It should be remarked that in establishing Laws of Iterated Logarithms, our improvements are achieved by combining the results on non-uniform estimates in the Central Limit Theorem with Theorem A. The arguments appear to be simple.

2. MAIN RESULTS

The following result deals with non-uniform estimates in the Central Limit Theorem for sums of independent random variables. A proof can be given following ideas in Katz (1963) and Bikelis (1966).

PROPOSITION 2.1. Let $g \in G$. Let $X_1, X_2, \cdots, X_n$ be independent random variables such that $EX_i = 0$ and $E(X_i^2 g(X_i)) < \infty$ for every i. Then for every x,

$$|F_n(x) - \Phi(x)| \leq A(1+|x|)^{-2}(s_n^2 g((1+|x|)s_n))^{-1} \sum_{i=1}^{n} E(X_i^2 g(X_i)), \qquad (2.1)$$

where $A > 0$ is an absolute constant.

Wittmann (1985) obtained the following general result on the law of the iterated logarithm.

THEOREM B. Let X_n, $n\geq 1$ be a sequence of independent zero mean real random variables with $EX_n^2 < \infty$ for every $n\geq 1$ and $s_n^2 = \sum_{i=1}^n EX_i^2$, $n\geq 1$. If, for some $0 < \alpha \leq 1$,

$$\sum_{n\geq 1}(s_n^2 \; L_2 s_n^2)^{-(2+\alpha)/2} \; E|X_n|^{2+\alpha} < \infty$$

and

$$\lim_{n\to\infty} s_n = \infty, \quad \limsup_{n\to\infty} (s_{n+1}/s_n) < \infty, \tag{2.2}$$

then

$$\limsup_{n\to\infty} S_n/[s_n(2L_2s_n^2)^{1/2}] = 1 \text{ a.s.},$$

and

$$\liminf_{n\to\infty} S_n/[s_n(2L_2s_n^2)^{1/2}] = -1 \text{ a.s.}$$

In the proof of the above theorem, Wittmann (1985) used the direct estimates in the central limit theorem due to Butzer and Hahn (1978). Wittmann (1987) generalized Theorem B covering every $\alpha > 0$. Combining Proposition 2.1 and Theorem A, we will generalize Theorem B in a different direction. Our proof seems to be simpler than that given by Wittmann (1985). We would like to emphasize that the generalization of Theorem B by Wittmann (1987) does not imply our generalization of Theorem B.

THEOREM 2.2. Let X_n, $n\geq 1$ be a sequence of independent zero mean real random variables with $EX_n^2 < \infty$ for every $n\geq 1$ and $s_n^2 = \sum_{i=1}^n EX_i^2$, $n\geq 1$. If (2.2) holds and

$$\sum_{n\geq 1} E(X_n^2 \; g(X_n))/[a_n^2 \; g(a_n)] < \infty \text{ for some } g \in G, \tag{2.3}$$

where $a_n = s_n(L_2s_n^2)^{1/2}$, then

$$\limsup_{n\to\infty} S_n/[s_n(2L_2s_n^2)^{1/2}] = 1 \text{ a.s.},$$

and

$$\liminf_{n\to\infty} S_n/[s_n(2L_2s_n^2)^{1/2}] = -1 \text{ a.s.}$$

PROOF. Using Proposition 2.1 and (2.2), we have that for $|x| \geq 1$,

$$\sum_{n\geq 1} \min\{1, (a_{n+1}-a_n)/a_n\} \; |P(S_n < xs_n(L_2 s_n^2)^{1/2}) - \Phi(x(L_2 s_n^2)^{1/2})|$$

$$\leq C_1 \sum_{n\geq 1}[(s_{n+1}^2 - s_n^2)/s_n^2] \sum_{i=1}^{n} E(X_i^2 g(X_i))/[a_n^2 g(a_n)]$$

$$\leq C_1 \sum_{i\geq 1}(\sum_{n\geq i}[s_{n+1}^2 - s_n^2]/[s_n^4 L_2 s_n^2 g(a_n)]) \; E(X_i^2 g(X_i))$$

$$\leq C_1 \sum_{i\geq 1}(\sum_{n\geq i}[s_{n+1}^2 - s_n^2]/s_n^4) \; E(X_i^2 g(X_i))/[(L_2 s_i^2) g(a_i)]$$

$$\leq C_2 \sum_{i\geq 1}(\int_{s_i^2}^{\infty} dx/x^2) \; E(X_i^2 g(X_i))/[(L_2 s_i^2) g(a_i)]$$

$$= C_2 \sum_{i\geq 1} E(X_i^2 g(X_i))/[a_i^2 g(a_i)] < \infty, \text{ by (2.3)}$$

Consequently, if $x \geq 1$, then

$$\sum_{n\geq 1} \min\{1, (a_{n+1}-a_n)/a_n\} \; P(S_n > a_n x) < \infty \tag{2.4}$$

if and only if

$$\sum_{n\geq 1} \min\{1, (a_{n+1}-a_n)/a_n\} \; (1 - \Phi(x(L_2 s_n^2)^{1/2}) < \infty. \tag{2.5}$$

From (2.2), it is easy to see that (2.5) is equivalent to

$$\sum_{n\geq 1}[(s_{n+1}^2 - s_n^2)/s_n^2] \; (1 - \Phi(x(L_2 s_n^2)^{1/2}) < \infty. \tag{2.6}$$

Observe that as $n \to \infty$

$$1 - \Phi(x(L_2 s_n^2)^{1/2}) \sim (2\pi)^{-1/2} \exp\{-x^2 L_2 s_n^2/2\} \; (x(L_2 s_n^2)^{1/2})^{-1}$$

$$= (2\pi)^{-1/2} \; (Ls_n^2)^{-x^2/2} \; (x(L_2 s_n^2)^{1/2})^{-1}.$$

Thus for $x \geq 1$, (2.6) is equivalent to

$$\sum_{n\geq 1}[(s_{n+1}^2 - s_n^2)/s_n^2] \; (Ls_n^2)^{-x^2/2} \; (L_2 s_n^2)^{-1/2} < \infty.$$

Note that

$$\int_1^{\infty} t^{-1} \; (Lt)^{-x^2/2} \; (L_2 t)^{-1/2} \, dt \; < \infty \quad \text{for } x > \sqrt{2},$$

$$= \infty \quad \text{for } x \leq \sqrt{2}$$

implies

$$\sum_{n\geq 1}[(s_{n+1}^2 - s_n^2)/s_n^2]\ (Ls_n^2)^{-x^2/2}\ (L_2s_n^2)^{-1/2} \quad < \infty \quad \text{for } x > \sqrt{2}$$

$$= \infty \quad \text{for } x \leq \sqrt{2}.$$

Therefore

$$\sum_{n\geq 1}\min\{1,\ (a_{n+1}-a_n)/a_n\}\ P(S_n > a_nx) \quad < \infty \quad \text{for } x > \sqrt{2}$$

$$= \infty \quad \text{for } x \leq \sqrt{2}.$$

By Theorem A,

$$\lim\sup_{n\to\infty} S_n/[s_n(2L_2s_n^2)^{1/2}] = 1 \quad \text{a.s.},$$

and

$$\lim\inf_{n\to\infty} S_n/[s_n(2L_2s_n^2)^{1/2}] = -1 \quad \text{a.s.},$$

by replacing X_i by $-X_i$.

Looking at the proof of Theorem 2.2, in fact, we have virtually established the following result. The reason that Theorem 2.2 is recast as Theorem 2.3 is that it becomes transperent that the conditions of Theorem 2.3 are weaker than the conditions of Theorem 3 of Petrov (1975), p.305 and the same conclusion holds.

THEOREM 2.3. Let X_n, $n \geq 1$ be a sequence of independent zero mean real random variables with $EX_n^2 < \infty$ for every $n \geq 1$ and $s_n^2 = \sum_{i=1}^n EX_i^2$, $n \geq 1$. If $s_n \to \infty$ as $n \to \infty$, $\limsup_{n\to\infty} s_{n+1}/s_n < \infty$, and for all x such that $|x| \geq 1$,

$$\sum_{n\geq 1}[(s_{n+1}^2-s_n^2)/s_n^2]\ |P(S_n/s_n < x(L_2s_n^2)^{1/2}) - \Phi(x(L_2s_n^2)^{1/2})| < \infty,$$

then

$$\limsup_{n\to\infty} S_n/[s_n(2L_2s_n^2)^{1/2}] = 1 \quad \text{a.s.},$$

and

$$\liminf_{n\to\infty} S_n/[s_n(2L_2s_n^2)^{1/2}] = -1 \quad \text{a.s.}$$

REMARKS. In order to show that Theorem B of Wittmann (1985) is a special case of Theorem 2.2 above, one merely needs to take the function g in Theorem 2.2 to be: $g(x) = |x|^\alpha$, $x \in \mathbb{R}$. One should also note that with respect to Theorem 2.2 that Condition (2.3) requires more than finite second moments of the random variables in order for it to hold.

We now give an example for which the conditions of Theorem 2.2 (thus of Theorem 2.3) are satisfied, but neither the conditions of Theorem 1 of Teicher (1974) nor the conditions of Theorem 1.1 of Tomkins (1983) are satisfied.

EXAMPLE 1. Define a sequence X_n, $n \geq 1$ of independent random variables as follows: for n = 1,2,3,

$$P(X_n = \pm 1) = 1/2,$$

and for $n \geq 4$,

$$P(X_n = \pm 1) = 1/2 - (2Ln)^{-1};$$

$$P(X_n = \pm n^{1/2}) = (2nLn)^{-1};$$

$$P(X_n = 0) = (Ln)^{-1} - (nLn)^{-1}.$$

Clearly, $EX_n = 0$ and $EX_n^2 = 1$ for all n. Consequently, $s_n^2 = n$ for all n. Note that, for $n \geq 4$,

$$E|X_n|^3 = 1 - (Ln)^{-1} + n^{1/2}(Ln)^{-1},$$

and thus,

$$\sum_{n\geq 1} (nL_2n)^{-3/2}E|X_n|^3 < \infty.$$

Therefore, by Theorem 2.2,

$$\limsup_{n\to\infty} S_n/(2nL_2n)^{1/2} = 1 \quad \text{a.s.},$$

and

$$\liminf_{n\to\infty} S_n/(2nL_2n)^{1/2} = -1 \quad \text{a.s.}$$

However, since

$$\sum_{n\geq 3} (nLn)^{-1} = \infty,$$

and

$$\sum_{n\geq 3} (nL_3n)^{-1} = \infty,$$

it can be verified that for every $\epsilon > 0$ and $\delta > 0$,

$$\sum_{n\geq 1} P(|X_n| > \epsilon n^{1/2}(L_2n)^{-1/2}) = \infty,$$

and

$$\sum_{n\geq 1} (nL_2n)^{-1}E(X_n^2 I(\epsilon n^{1/2}(L_2n)^{-1/2} < |X_n| \leq \delta(nL_2n)^{1/2})) = \infty.$$

Thus Theorem 1 of Teicher (1974) and Theorem 1.1 of Tomkins (1983) are not applicable here.

The next example satisfies the conditions of Kolmogorov's Law of the Iterated Logarithm (thus of Theorem 1 of Teicher (1974) and of Theorem 1.1 of Tomkins (1983)), but not those of Theorem 2.2 above.

EXAMPLE 2. Let X_n, $n \geq 1$ be a sequence of independent random variables such that for each $n \geq 1$,

$$P(X_n = 0) = 1 - n^{-1}(L_2n)(L_3n),$$

and

$$P(X_n = \pm n^{1/2}(L_2n)^{-1/2}(L_3n)^{-1/2}) = (2n)^{-1}(L_2n)(L_3n).$$

Clearly, $EX_n = 0$, $EX_n^2 = 1$, and $s_n^2 = n$ for all n. Further, X_n, $n \geq 1$ satisfies the conditions of Kolmogorov's Law of the Iterated Logarithm. Thus we have

$$\limsup_{n\to\infty} S_n/(2nL_2n)^{1/2} = 1 \quad \text{a.s.},$$

and

$$\liminf_{n\to\infty} S_n/(2nL_2n)^{1/2} = -1 \quad \text{a.s.}$$

However, for any $g \in G$,

$$\sum_{n\geq 1} E[X_n^2 g(X_n)]/[nL_2ng((nL_2n)^{1/2})]$$

$$= \sum_{n\geq 1} [n(L_2n)^{-1}(L_3n)^{-1}g((n/((L_2n)(L_3n)))^{1/2})]/[nL_2ng((nL_2n)^{1/2})]$$

$$\geq \sum_{n\geq 1} n^{-1}(L_2n)^{-2}(L_3n)^{-1/2} = \infty \quad (\text{since } x/g(x) \uparrow).$$

Thus Theorem 2.2 is not applicable here.

Although Kolmogorov's Law of the Iterated Logarithm (thus Theorem 1 of Teicher (1974) and Theorem 1.1 of Tomkins (1983)) does not follow from Theorem 2.2 above, we conjecture that Theorem 1 of Teicher (1974) and Theorem 1.1 of Tomkins (1983) (thus Kolmogorov's Law of the Iterated Logarithm) follow from Theorem 2.3 above. In fact, the classical Hartman-Wintner Law of the Iterated Logarithm (1941) follows from our work. Let X, X_n, $n\geq 1$ be a sequence of independent identically distributed random variables such that $EX = 0$ and $EX^2 = 1$. Set

$$\sigma_n^2 = E(X^2 I(|X|<n^{1/2})) - (E(XI(|X|<n^{1/2})))^2, \ n \geq 1.$$

By Friedman-Katz-Koopmans Theorem (1966),

$$\sum_{n\geq 1} n^{-1} \sup_x |P(\sigma_n^{-1} n^{-1/2} \sum_{j=1}^n X_j < x) - \Phi(x)| < \infty.$$

Note that $\sigma_n \to 1$ as $n \to \infty$. By Theorem 2.3, it follows that

$$\limsup_{n\to\infty} S_n/(2nL_2 n)^{1/2} = 1 \text{ a.s.},$$

and

$$\liminf_{n\to\infty} S_n/(2nL_2 n)^{1/2} = -1 \text{ a.s.}$$

Let X_n, $n\geq 1$ be a sequence of independent zero mean random variables, $g \in G$, and a_n, $n\geq 1$ a sequence of positive numbers such that $a_n \uparrow \infty$ as $n \to \infty$. Using Theorem 9 of Petrov (1975), p.267 and Kronecker's lemma, one can show that

$$\sum_{n\geq 1} E[|X_n| g(X_n)]/[a_n g(a_n)] < \infty \tag{2.7}$$

implies

$$S_n/a_n \to 0 \text{ a.s. as } n \to \infty. \tag{2.8}$$

Theorem 2.2 shows that if (2.2) holds and $a_n = s_n(L_2 s_n^2)^{1/2}$, $n\geq 1$, then

$$\sum_{n\geq 1} E[|X_n|^2 g(X_n)]/[a_n^2 g(a_n)] < \infty \tag{2.9}$$

implies that X_n, $n\geq 1$ obeys the Law of the Iterated Logarithm. Note that conditions (2.7) and (2.9) do not imply each other. We now give a result whose conclusion is in the spirit of (2.8) under a condition which is in the spirit of (2.9). This result is a consequence of Proposition 2.1 and Theorem A.

THEOREM 2.4. Let X_n, $n\geq 1$ be a sequence of independent zero mean random variables with $EX_n^2 < \infty$ for every $n\geq 1$ and $s_n^2 = \sum_{i=1}^n EX_i^2$, $n\geq 1$. Let a_n, $n\geq 1$ be a sequence of positive numbers and $g \in G$. Suppose that

$$a_n/s_n \uparrow, \quad \limsup_{n\to\infty} a_{n+1}/a_n < \infty, \tag{2.10}$$

$$\lim_{n\to\infty} s_n(L_2 s_n^2)^{1/2}/a_n = 0, \tag{2.11}$$

and

$$\sum_{n\geq 1} [EX_n^2 g(X_n)]/[a_n^2 g(a_n)] < \infty. \tag{2.12}$$

Then

$$S_n/a_n \to 0 \text{ a.s. as } n \to \infty.$$

PROOF. It follows from Proposition 2.1 that for every $x \neq 0$

$$|P(S_n/s_n < xa_n/s_n) - \Phi(xa_n/s_n)|$$
$$\leq A[\textstyle\sum_{i=1}^{n} EX_i^2 g(X_i)]/[a_n^2 g(a_n)],$$

where $A > 0$ is a constant depending only on x. Using (2.10), we have that for $x \neq 0$,

$$\sum_{n\geq 1}\min\{1, (a_{n+1}-a_n)/a_n\} |P(S_n < xa_n) - \Phi(xa_n/s_n)|$$
$$\leq A \sum_{n\geq 1}[(a_{n+1}-a_n)/a_n][\textstyle\sum_{i=1}^{n} EX_i^2 g(X_i)]/[a_n^2 g(a_n)]$$
$$= A \sum_{i\geq 1}\{\textstyle\sum_{n\geq i}(a_{n+1}-a_n)/[a_n^3 g(a_n)]\} EX_i^2 g(X_i)$$
$$\leq C \sum_{i\geq 1}(\textstyle\int_{a_i}^{\infty} [t^3 g(a_i)]^{-1} dt) EX_i^2 g(X_i)$$
$$\leq C \sum_{i\geq 1}[EX_i^2 g(X_i)]/[a_i^2 g(a_i)] < \infty,$$

where C is also a constant depending only on x. Therefore, for every $\epsilon > 0$

$$\sum_{n\geq 1} \min\{1, (a_{n+1}-a_n)/a_n\} P(|S_n| \geq \epsilon a_n) < \infty \tag{2.13}$$

if and only if

$$\sum_{n\geq 1} \min\{1, (a_{n+1}-a_n)/a_n\} \Phi(-\epsilon a_n/s_n) < \infty.$$

Observe that as $n \to \infty$

$$\Phi(-\epsilon a_n/s_n) = o(\exp\{-\epsilon^2 a_n{}^2/2s_n^2\}).$$

Since $t/L_2 t$ is monotonically increasing on $(0,\infty)$, then $[s_n^2 L_2 s_n^2]/a_n{}^2 \to 0$ as $n \to \infty$ if and only if $[s_n^2 L_2 s_n^2]/[(a_n^2/L_2 a_n^2)L_2(a_n^2/L_2 a_n^2)] \to 0$ as $n \to \infty$. Consequently, (2.11) implies $[s_n^2 L_2 a_n]/a_n^2 \to 0$, i.e., $s_n^2/a_n^2 = o(1/L_2 s_n^2)$ as $n \to \infty$. It now follows that

$$\exp\{-\epsilon^2 a_n{}^2/2s_n^2\} = o(\exp\{-2L_2 a_n\}) \text{ as } n \to \infty.$$

From (2.10),

$$\sum_{n\geq 1}[(a_{n+1}-a_n)/a_n] \exp\{-2L_2 a_n\}$$
$$\leq \sum_{n\geq 1}(a_{n+1}-a_n)/[a_n(La_n)^2]$$
$$\leq C_1 \int_{a_1}^{\infty}[t(Lt)^2]^{-1} dt < \infty.$$

Now we have that (2.13) holds for every $\epsilon > 0$. It is easy to see that

$$\sum_{n\geq 1} \min\{1, (a_{n+1}-a_n)/a_n\} P(|S_n| \geq \epsilon a_n) < \infty \quad \text{for every} \quad \epsilon > 0.$$

From Theorem A,

$$S_n/a_n \to 0 \text{ a.s. as } n \to \infty.$$

REMARK. If Condition (2.11) is replaced by

$$\limsup_{n\to\infty} s_n(L_2 s_n^2)^{1/2} /a_n = \Lambda < \infty,$$

one can show by a similar argument that

$$\limsup_{n\to\infty} |S_n|/a_n \leq \sqrt{2}\, \Lambda \quad \text{a.s.}$$

When a sequence X_n, $n\geq 1$ is stochastically dominated by a nonnegative random variable X, Egorov (1971) gave the following sufficient conditions for X_n, $n\geq 1$ to obey the Law of the Iterated Logarithm.

THEOREM C. Let X_n, $n\geq 1$ be a sequence of independent zero mean random variables with $EX_n^2 < \infty$ for all $n\geq 1$ and $s_n^2 = \sum_{i=1}^n EX_i^2$, $n\geq 1$. Suppose there is a nonnegative random variable X such that for sufficiently large n and x

$$(1/n)\sum_{i=1}^n P(|X_i| > x) \leq P(X > x). \tag{2.14}$$

If

$$EX^2 < \infty \tag{2.15}$$

and

$$\liminf_{n\to\infty} s_n^2/n > 0, \tag{2.16}$$

then

$$\limsup_{n\to\infty} S_n/[s_n(2L_2 s_n^2)^{1/2}] = 1 \quad \text{a.s.},$$

and

$$\liminf_{n\to\infty} S_n/[s_n(2L_2 s_n^2)^{1/2}] = -1 \quad \text{a.s.}$$

As a last application of Proposition 2.1, we shall extend the above theorem.

THEOREM 2.5. Let X_n, $n\geq1$ be a sequence of independent zero mean random variables with $EX_n^2 < \infty$ for all $n\geq1$ and $s_n^2 = \sum_{i=1}^n EX_i^2$, $n\geq1$. Suppose there exists a nonnegative random variable X such that for all sufficiently large n and x

$$(1/n)\sum_{i=1}^n P(|X_i| > x) \leq P(X > x). \tag{2.17}$$

If

$$E(X^2/L_2X) < \infty \tag{2.18}$$

and

$$0 < \liminf_{n\to\infty} s_n^2/n \leq \limsup_{n\to\infty} s_n^2/n < \infty, \tag{2.19}$$

then

$$\limsup_{n\to\infty} |S_n|/[s_n(2L_2s_n^2)^{1/2}] \leq 1 \quad \text{a.s.} \tag{2.20}$$

Moreover, if the following additional condition is satisfied

$$\lim_{n\to\infty} (1/n)\sum_{i=1}^n E(X_i^2\, I(|X_i| \geq \sqrt{iL_2 i})) = 0, \tag{2.21}$$

then

$$\limsup_{n\to\infty} S_n/[s_n(2L_2s_n^2)^{1/2}] = 1 \quad \text{a.s.},$$

and

$$\liminf_{n\to\infty} S_n/[s_n(2L_2s_n^2)^{1/2}] = -1 \quad \text{a.s.}$$

PROOF. For each $n\geq1$, let

$$Y_n = X_n\, I(|X_n| < \sqrt{n\, L_2 n}), \qquad Z_n = X_n - Y_n,$$

$$U_n = Y_n - EY_n, \qquad V_n = Z_n - EZ_n, \qquad s_n'^2 = \sum_{i=1}^n EU_n^2.$$

We first prove that from (2.17) and (2.19)

$$(\sum_{i=1}^n V_i)/\sqrt{2n\, L_2 n} \to 0 \quad \text{a.s.} \quad \text{as} \quad n\to\infty. \tag{2.22}$$

Observe that if N is large enough (assume $N = 2^{m-1}$ for some $m\geq1$)

$$\begin{aligned}\sum_{n\geq N} P(Z_n \neq 0) &= \sum_{n\geq N} P(|X_n| \geq \sqrt{n\, L_2 n}) \\ &\leq \sum_{k\geq m} \sum_{n=2^{k-1}}^{2^k} P(|X_n| \geq \sqrt{n\, L_2 n}) \\ &\leq \sum_{k\geq m} 2^k\, P(X \geq (2^{k-1}\, L_2 2^{k-1})^{1/2})\end{aligned}$$

$$\le 4 \int_0^\infty P(X^2 \ge x\, L_2 x)\, dx$$

$$\le 4 \int_0^\infty P(X^2/L_2 X \ge (1/2)x)\, dx$$

$$= 16\, E(X^2/L_2 X) < \infty.$$

It follows that

$$\sum_{i=1}^n Z_i/\sqrt{2n\, L_2 n} \to 0 \text{ a.s. as } n \to \infty. \tag{2.23}$$

Note that (2.19) and $|EZ_n|/EX_n^2 \le [EX_n^2/\sqrt{n\, L_2 n}]/EX_n^2 \to 0$ as $n \to \infty$ imply

$$\sum_{i=1}^n EZ_i/s_n^2 \to 0 \text{ as } n \to \infty. \tag{2.24}$$

Now (2.23), (2.24) and (2.19) show that (2.22) holds which is equivalent to

$$\sum_{i=1}^n V_i/[s_n(2L_2 s_n^2)^{1/2}] \to 0 \text{ a.s. as } n \to \infty, \tag{2.25}$$

by (2.19). To complete the proof of (2.20), we only need to prove

$$\sum_{n\ge 1} E|U_n|^3/(nL_2 n)^{3/2} < \infty, \tag{2.26}$$

in view of (2.19), $s_n'/s_n \le 1$ and Remark. If N is large enough

$$\sum_{n\ge N} E|U_n|^3/(n\, L_2 n)^{3/2} \le 24 \sum_{n\ge N}[\int_0^{\sqrt{nL_2 n}} t^2\, P(|X_n| \ge t)\, dt]/(n\, L_2 n)^{3/2}$$

$$\le 24 \sum_{n\ge N}[\int_0^s t^2\, dt]/(n\, L_2 n)^{3/2}$$

$$+ 24 \sum_{n\ge N}[\int_s^{\sqrt{nL_2 n}} t^2\, P(|X_n| \ge t)\, dt]/(n\, L_2 n)^{3/2},$$

where s is a fixed number large enough. Obviously, the first series converges. By (2.17) and (2.18), we estimate the second series by following

$$\sum_{n\ge N}[\int_s^{\sqrt{nL_2 n}} t^2\, P(|X_n| \ge t)\, dt]/(n\, L_2 n)^{3/2}$$

$$\le \sum_{k\ge m}[\int_s^{(2^k\, L_2 2^k)^{1/2}} t^2 \sum_{i=2^{k-1}}^{2^k} P(|X_i| \ge t)\, dt]/(2^{k-1}\, L_2 2^{k-1})^{3/2},$$

for some suitable m,

$$\le 8 \sum_{k\ge m}[\int_0^{(2^k L_2 2^k)^{1/2}} t^2\, 2^k\, P(X \ge t)\, dt]/(2^k L_2 2^k)^{3/2}$$

$$= 8 \sum_{k\geq m} \sum_{i=1}^{k}[\int_{b_{i-1}}^{b_i} t^2\, 2^k\, P(X \geq t)\, dt](2^k L_2 2^k)^{3/2}$$

$$\leq 8 \sum_{k\geq m} \sum_{i=1}^{k}[b_i^3\, 2^k\, P(X \geq b_{i-1})]/(2^k L_2 2^k)^{3/2}$$

$$\leq 8 \sum_{i\geq 1}[\sum_{k\geq 1} 2^{-k/2}\, (L_2 2^k)^{-3/2}]\, b_i^3\, P(X \geq b_{i-1})$$

$$\leq 32 \sum_{i\geq 1} 2^{-i/2}\, (L_2 2^i)^{-3/2}\, b_i^3\, P(X \geq b_{i-1})$$

$$= 64[\sum_{i\geq 1} 2^{i-1}\, P(X \geq (2^{i-1}\, L_2 2^{i-1})^{1/2}) + 1] < \infty.$$

We conclude that (2.20) holds.

Under the conditions (2.19) and (2.21), it is easy to verify that $\lim_{n\to\infty} s_n'/s_n = 1$, and further, it follows from (2.25), (2.26) and Theorem 2.2 that

$$\limsup_{n\to\infty} S_n/[s_n(2L_2 s_n^2)^{1/2}] = 1 \quad \text{a.s.},$$

and

$$\liminf_{n\to\infty} S_n/[s_n(2L_2 s_n^2)^{1/2}] = -1 \quad \text{a.s.}$$

In view of the characterization of the law of the iterated logarithm in Banach spaces given in Ledoux and Talagrand (1988), the condition (2.18) merits special interest. Note that the conditions (2.18) and (2.21) both follow from the condition (2.15) under which Theorem C operates. We shall give an example which fails to meet the conditions of Theorem C but Theorem 2.5 is applicable.

EXAMPLE 3. Let $p_n = \exp\{e^{n^2}\}$, $n\geq 1$ and X_n, $n\geq 1$ a sequence of independent random variables with distributions given as follows.

$$\begin{aligned} P(X_n = x) &= (2p_n\, L_4 p_n)^{-1}, && \text{if } x = \pm\sqrt{p_n}, \\ &= (1/2)(1 - 1/L_4 p_n), && \text{if } x = \pm 1, \\ &= (1/L_4 p_n)(1 - 1/p_n), && \text{if } x = 0, \quad \text{for all } n\geq 1. \end{aligned}$$

It is easy to see that $EX_n = 0$ and $EX_n^2 = 1$ for all $n\geq 1$. Let

$$G_1(x) = \sup_{n\geq 1} (1/n) \sum_{i=1}^n P(|X_i| \geq x)$$

and

$$G_2(x) = \sup_{n\geq 1} P(|X_n| \geq x), \quad -\infty < x < \infty.$$

Obviously, $G_1(x) \leq G_2(x)$ for all x and $G_2(x) \to 0$ as $x \to \infty$. Consequently, both $1 - G_1$ and $1 - G_2$ are distribution functions. We first show that

$$\int_0^\infty x\, G_1(x)\, dx = \infty. \tag{2.27}$$

In fact,

$$\int_0^\infty x\, G_1(x)\, dx \geq \int_0^\infty x\, [\sup_{n\geq 1} (1/n)P(|X_n| \geq x)]\, dx$$

$$\geq \sum_{k\geq 1} \int_{\sqrt{p_k}}^{\sqrt{p_{k+1}}} x\, [\sup_{n\geq 1} (1/n)P(|X_n| \geq x)]\, dx$$

$$\geq \sum_{k\geq 1} \int_{\sqrt{p_k}}^{\sqrt{p_{k+1}}} (x/(k+1))\, P(|X_{k+1}| \geq x)\, dx$$

$$= \sum_{k\geq 1}[(k+1)\, p_{k+1}\, L_4 p_{k+1}]^{-1} \int_{\sqrt{p_k}}^{\sqrt{p_{k+1}}} x\, dx$$

$$\geq (1/4) \sum_{k\geq 1} 1/[(k+1)\, L_2(k+1)] = \infty.$$

Note that (2.27) implies that (2.15) fails. For, if any nonnegative random variable Y satisfies

$$(1/n) \sum_{i=1}^n P(|X_i| \geq x) \leq P(Y \geq x) \quad \text{for all } x \geq 0 \text{ and } n \geq 1,$$

then from the definition of $G_1(x)$

$$G_1(x) \leq P(Y \geq x) \quad \text{for all } x \geq 0$$

and from (2.27) we have that $EY^2 = \infty$. Now we show that

$$\int_0^\infty (x/L_2 x)\, G_1(x)\, dx < \infty. \tag{2.28}$$

In fact,

$$\int_0^\infty (x/L_2 x)\, G_1(x)\, dx \leq 2 \int_0^\infty (x/L_2 x^2)\, G_2(x)\, dx$$

$$= 2[\int_0^2 (x/L_2 x^2)\, G_2(x)\, dx + \int_2^{\sqrt{p_1}} (x/L_2 x^2)\, G_2(x)\, dx$$

$$+ \sum_{n\geq 1} \int_{\sqrt{p_n}}^{\sqrt{p_{n+1}}} (x/L_2x^2)\ G_2(x)\ dx]$$

$$\leq 2[2 + p_1 + \sum_{n\geq 1} (p_{n+1}/L_2p_n)\ (1/(p_{n+1}\ L_4p_{n+1}))]$$

$$\leq 2[2 + p_1 + \sum_{n\geq 1} (1/n^2)] < \infty.$$

But (2.28) implies that the condition (2.18) holds where X is a random variable with the distribution function $1 - G_1$. Moreover, as $n \to \infty$

$$EX_n^2\ I(|X_n| \geq \sqrt{nL_2n}) = 1/L_4p_n \to 0.$$

It now follows that

$$\lim_{n\to\infty} (1/n) \sum_{i=1}^n EX_i^2\ I(|X_i| \geq (iL_2i)^{1/2}) = 0.$$

By Theorem 2.5

$$\limsup_{n\to\infty} S_n/\sqrt{2n\ L_2n} = 1 \quad \text{a.s.},$$

and

$$\liminf_{n\to\infty} S_n/\sqrt{2n\ L_2n} = -1 \quad \text{a.s.}$$

The following example shows that for a sequence X_n, $n\geq 1$ to obey the Law of the Iterated Logarithm, i.e.,

$$\limsup_{n\to\infty} S_n/[s_n(2L_2s_n^2)^{1/2}] = 1 \quad \text{a.s.},$$

and

$$\liminf_{n\to\infty} S_n/[s_n(2L_2s_n^2)^{1/2}] = -1 \quad \text{a.s.}$$

in the framework of Theorem 2.5, the condition (2.21) is optimal in a way.

EXAMPLE 4. Let $p_n = \exp\{e^{n^2}\}$, $n \geq 1$, and $0 \leq a < 1$. Let X_n, $n\geq 1$ be a sequence of independent random variables with distributions

$$\begin{aligned} P(X_n = x) &= (1-a)/2p_n && \text{for } x = \pm\sqrt{p_n}, \\ &= a/2 && \text{for } x = \pm 1, \\ &= (1-a)(1-1/p_n) && \text{for } x = 0, \text{ for each } n \geq 1. \end{aligned}$$

It is easy to see that $EX_n = 0$ and $EX_n^2 = 1$ for all $n \geq 1$. Let

$$H(x) = \sup_{n\geq 1} P(|X_n| \geq x), \quad x \geq 0.$$

We now have that

$$\int_0^\infty (x/L_2x)\ H(x)\ dx < \infty$$

and

$$(1/n)\ \sum_{i=1}^n EX_i^2\ I(|X_i| \geq (iL_2i)^{1/2}) = 1/n + ((n-1)/n)(1-a) \rightarrow 1-a$$

as $n \to \infty$. Consequently,

$$\limsup_{n\to\infty} S_n/\sqrt{2n\ L_2n} = a \quad a.s.$$

and

$$\liminf_{n\to\infty} S_n/\sqrt{2n\ L_2n} = -a \quad a.s.$$

From Examples 3 and 4, we know that the conditions in Theorem 2.5 are, in some sense, not only optimal, but also the value of the limit supremum of $S_n/[s_n(2L_2s_n^2)^{1/2}]$ depends on the value of the limit of $s_n^{-2}\sum_{i=1}^n E(X_i^2 I(|X_i| \geq (iL_2i)^{1/2}))$.

The referee has pointed out the following. Theorem A, which plays a crucial role in the establishment of Theorem 2.2, obtains for more general sequences a_n, $n\geq1$ satisfying only $a_n/s_n \uparrow \infty$. But Theorem 2.2 was proved for the particular choice $a_n = s_n(L_2s_n^2)^{1/2}$, $n\geq1$ of weights. Can there be a version of Theorem 2.2 for general sequences a_n, $n\geq1$?

ACKNOWLEDGMENTS. The authors are delighted to acknowledge the meticulous care the referee has taken in scrutinizing our paper. The referee has brought to our attention the works of Teicher and Tomkins which also deal with the general theme of this paper. A careful review of the papers by Teicher and Tomkins led us to make a critical comparison of our work with their work and thus enhancing the value of our work. Mr. Deli Li gratefully acknowledges the support of Youth Science Foundation of China for his research. Professor Wang gratefully acknowledges the support of the National Natural Science Foundation of China for his research.

REFERENCES

1. BIKELIS, A. Ocenki ostatočnogo člena v central'noĭ predel'noĭ teoreme. Litovskiĭ matem.sb., 6 (1966), No.3, 323-346.
2. BUTZER, P.L. and HAHN, L. General Theorems for Rates of Convergence in Distribution of Random Variables, I : General Limit Theorems, Journal of Multivariate Anal., 8 (1978), 181-208.
3. EGOROV, V.A. Obobscenie teoremy Hartmana-Vintnera o zakone povtornogo logarifma, Vestnik Leningrad.Univ., (1971), No.7, 22-28.
4. FELLER, W. On the Oscillations of Sums of Independent Random Variables, Ann.Math., 91 (1970), No.2, 402-418.
5. FRIEDMAN, N., KATZ, M. and KOOPMANS, L.H. Convergence Rates for the Central Limit Theorem, Proc.Nat.Acad.Sci.,USA, 56 (1966), 1062-1065.
6. HARTMAN, P. and WINTNER, A. On the Law of the Iterated Logarithm, Amer. J.Math., 63 (1941), No.1, 169-176.
7. KATZ, M. Note on the Berry-Esseen Theorem, Ann.Math.Stat., 34 (1963), No.3, 1107-1108.
8. LEDOUX, M. and TALAGRAND, M. Characterization of the Law of the Iterated Logarithm in Banach Spaces, Ann.Probab., 16 (1988), No.3, 1242-1264.
9. PETROV, V.V. Sums of Independent Random Variables, Springer-Verlag, New York, 1975.
10. TEICHER, H. On the Law of the Iterated Logarithm, Ann.Probab., 2 (1974), 714-728.
11. TOMKINS, R.J. Lindeberg Functions and the Law of the Iterated Logarithm, Z.Wahrscheinlichkeitstheorie Verw.Gebiete, 65 (1983), 135-143.
12. TOMKINS, R.J. A Generalized Law of the Iterated Logarithm, Statistics & Probability Letters, 10 (1990), 9-15.
13. WITTMANN, R. A General Law of Iterated Logarithm, Z.Wahrscheinlichkeitstheorie.Verw.Gebiete, 68 (1985), 521-543.
14. WITTMANN, R. Sufficient Moment and Truncated Moment Conditions for the Law of the Iterated Logarithm, Probab. Th. Rel. Fields, 75 (1987), 509-530.