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BOUNDEDNESS AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A FORCED DIFFERENCE EQUATION

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ABSTRACT. The authors consider the nonlinear difference equation

$$\Delta[y_n + p_n y_{n-h}] + q_n f(y_{n-k}) = r_n$$

where $\Delta y_n = y_{n+1} - y_n$, $\{p_n\}$, $\{q_n\}$, and $\{r_n\}$ are real sequences, and $uf(u) > 0$ for $u \neq 0$. Sufficient conditions for boundedness and convergence to zero of certain types of solutions are given. Examples illustrating the results are also included.

KEY WORDS AND PHRASES. Difference equations, nonlinear, forced, boundedness, asymptotic behavior.
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1. INTRODUCTION.

In this paper we obtain results on the asymptotic behavior of solutions of the forced nonlinear difference equation

$$\Delta[y_n + p_n y_{n-h}] + q_n f(y_{n-k}) = r_n \tag{E}$$

where $\Delta y_n = y_{n+1} - y_n, h, k \in \mathbb{N} = \{0, 1, \dots\}$, $\{p_n\}$, $\{q_n\}$, and $\{r_n\}$ are sequences of real numbers, and $f : \mathbb{R} \to \mathbb{R}$ is continuous with $uf(u) > 0$ for $u \neq 0$. A solution of (E) is a sequence $\{y_n\}$ defined for $n \geq N_0 - \max\{h, k\}$, $N_0 \geq 0$, which satisfies (E) for $n \geq N_0$. We will classify solutions of (E) by borrowing some terminology introduced in [5] for the solutions of differential equations. A nontrivial solution $\{y_n\}$ of (E) is said to be *oscillatory* if for every positive integer $N_1 \geq N_0$ there exists $n \geq N_1$ such that $y_n y_{n+1} < 0$; it will be called *nonoscillatory* if there exists a positive integer N_2 such that y_n has fixed sign for all $n > N_2$; and will be called a *Z-type* solution if there exists a positive integer N_3 such that y_n does not change sign for $n \geq N_3$ but $y_n = 0$ for arbitrarily large values of n.

REMARK. Notice that when $r_n \not\equiv 0$, (E) may have Z-type solutions. For example, the equation

$$\Delta y_n + y_n = 1 + \cos\left(\frac{n\pi}{2}\right)$$

has the solution

$$y_n = 1 + \sin\left(\frac{n\pi}{2}\right) \geq 0$$

for all $n \geq 1$.

Our interest here is in obtaining results on the convergence to zero of all the nonoscillatory solutions of (E). The Z-type solutions, even though they have arbitrarily large zeros, behave in many respects like the nonoscillatory solutions. As a consequence, our results include that type of solution as well. Very few results of this type are known for nonlinear difference equations, and essentially no such results are known for equations with a forcing term. Most of the results known to this point in time are sufficient conditions for the oscillation of solutions of unforced equations. Recent contributions in this direction can be found, for example, in [2 - 4, 6, 8, and 10] and in the references contained therein. For a discussion of basic notions on difference equations, we refer the reader to Kelley and Peterson [7] and Mickens [11]; for more advanced topics we refer to the monographs by Agarwal [1] and Lakshmikantham and Trigiante [9].

2. MAIN RESULTS.

In the remainder of this paper, we will let $w_n = y_n + p_n y_{n-h}$. Our first theorem gives sufficient conditions to ensure that certain types of solutions of (E) are bounded, and our final result gives conditions that imply these types of solutions tend on to zero as $n \to \infty$.

THEOREM 1. If

$$q_n \geq 0, \tag{1}$$

$$\sum_{i=N_0}^{\infty} r_i \text{ converges}, \tag{2}$$

and there exists a constant P_1 such that

$$-1 < P_1 \leq p_n, \tag{3}$$

then any nonoscillatory or Z-type solution of (E) is bounded.

PROOF. Let $\{y_n\}$ be either a nonoscillatory or Z-type solution of (E). For definiteness, assume that there exists $N_1 \geq N_0$ such that $y_n \geq 0$ for $n \geq \min\{N_1 - k, N_1 - h\}$. Then, from (E), we have

$$w_{n+1} + \sum_{i=N_1}^{n} q_i f(y_{i-k}) = \sum_{i=N_1}^{\infty} r_i + w_{N_1}. \tag{4}$$

Clearly, there exists a constant $L_0 \geq 0$ such that either

$$\sum_{i=N_1}^{\infty} q_i f(y_{i-k}) = L_0, \tag{5}$$

or

$$\sum_{i=N_1}^{\infty} q_i f(y_{i-k}) = \infty. \tag{6}$$

If (5) holds, then (2) and (4) imply that $w_n \to L_1$ for some constant L_1. If $L_1 < 0$, then eventually $w_n < 0$. Thus, eventually $p_n < 0$ since $y_n \geq 0$, say $p_n < 0$ for $n \geq N_2 \geq N_1$. Then (3) implies that $-1 < P_1 < 0$ and that

$$0 < y_n \leq -P_1 y_{n-h}$$

for $n \geq N_2 + h$. Iterating for each fixed $n \geq N_2 + h$, we have that

$$y_{n+lh} < (-P_1)^l y_n$$

for $l \geq 1$. Since $0 < -P_1 < 1$, for each fixed n, $y_{n+lh} \to 0$ as $l \to \infty$, which implies that $y_n \to 0$ as $n \to \infty$. Hence, in this case $\{y_n\}$ is not only bounded but also tends to zero as $n \to \infty$. If

$L_1 \geq 0$ and $\{y_n\}$ is not bounded, then there exists an increasing subsequence $\{y_{n_j}\}$ of $\{y_n\}$ with $y_{n_j} \to \infty$ as $j \to \infty$, $n_1 > N_1 + h$, $y_{n_j} > w_{n_j}$ for $j \geq 1$, and

$$y_{n_j} = \max\{y_m : n_1 \leq m \leq n_j\}.$$

But $y_{n_j} > w_{n_j}$ implies that $p_{n_j} < 0$ for $j \geq 1$, so $-1 < P_1 < 0$ and

$$(1 + P_1)y_{n_j} = y_{n_j} + P_1 y_{n_j} < w_{n_j}$$

contradicting $w_n \to L_1$ and $n \to \infty$. Thus, we have proved that $\{y_n\}$ is bounded in case (5) holds.

Now suppose (6) holds. Then (2) implies that $w_n \to -\infty$ as $n \to \infty$, so eventually $w_n < 0$. Then, as argued above for the case $L_1 < 0$, eventually $p_n < 0$ and $\{y_n\}$ is not only bounded but satisfies $y_n \to 0$ as $n \to \infty$.

The proof for $y_n \leq 0$ is similar and will be omitted.

THEOREM 2. Suppose that, in addition to (1) - (2),

$$\sum_{i=N_0}^{\infty} q_i = \infty, \tag{7}$$

and

$$f(u) \text{ is bounded away from zero when } u \text{ is bounded away from zero.} \tag{8}$$

If

$$p_n \to 0 \text{ as } n \to \infty, \tag{9}$$

then any nonoscillatory or Z-type solution $\{y_n\}$ of (E) satisfies $y_n \to 0$ as $n \to \infty$.

PROOF. Let $\{y_n\}$ be a nonoscillatory or Z-type solution of (E), say $y_n \geq 0$ for $n \geq N_1 - h - k$ where $N_1 \geq N_0$ is a positive integer. Observe that (9) implies that (3) eventually holds, so $\{y_n\}$ is bounded. From the proof of Theorem 1, either (6) holds, or (5) holds and $w_n \to L_1$ as $n \to \infty$ for some constant L_1. Furthermore, it was also shown in the proof of Theorem 1 that $y_n \to 0$ as $n \to \infty$ if either (6) holds or (5) holds with $L_1 < 0$. Hence, we only need consider the case when (5) holds and $L_1 \geq 0$.

Notice, first, that since (5) holds, (7) - (8) imply that

$$\liminf_{n\to\infty} y_n = 0. \tag{10}$$

Also, by Theorem 1, $\{y_n\}$ is bounded. Now suppose that $L_1 > 0$. Then there exists $N_2 \geq N_1$ such that $w_n > L_1/2$, or

$$y_n > L_1/2 - p_n y_{n-h}$$

for $n \geq N_2 - k - h$. It then follows from (9) and the boundedness of $\{y_n\}$ that eventually $y_n > L_1/4 > 0$ contradicting (10). Thus $L_1 = 0$, and (9) and the boundedness of $\{y_n\}$ imply that

$$y_n = w_n - p_n y_{n-h} \to 0$$

as $n \to \infty$. The proof is similar for the case $y_n \leq 0$.

REMARK. It is interesting to observe that a requirement analogous to the condition

$$\sum_{i=N_0}^{n} r_i \text{ converges} \tag{2}$$

is necessary in the hypotheses of Theorem 1. Furthermore, conditions similar to (2) and

$$\sum_{i=N_0}^{\infty} q_i \text{ diverges} \tag{7}$$

are needed in the hypotheses of Theorem 2. To demonstrate this, consider the following examples:

$$\Delta\left(y_n + \frac{y_{n-1}}{n^2}\right) + \frac{y_{n-1}}{n} = 2 - \frac{1}{n(n+1)}, \quad n \geq 1; \tag{E_1}$$

$$\Delta\left(y_n + \frac{y_{n-3}}{n-2}\right) + y_{n-1} = \frac{n}{n-1} - \frac{2(n^2-2n+3)}{n(n+1)(n-2)(n-3)}, \quad n \geq 4; \tag{E_2}$$

and

$$\Delta\left(y_n + \frac{y_{n-3}}{n-2}\right) + \frac{y_{n-1}}{n^2} = \frac{1}{n(n-1)} - \frac{2(n^2-2n+3)}{n(n+1)(n-2)(n-3)}, \quad n \geq 4. \tag{E_3}$$

Equations (E_1) and (E_2) satisfy all the hypotheses of Theorems 1 and 2 except condition (2). Notice that $\{y_n\} = \{n+1\}$ is an unbounded nonoscillatory solution of (E_1) while $\{y_n\} = \{\frac{n+1}{n}\}$ is a solution of (E_2) satisfying $y_n \to 1$ as $n \to \infty$. The latter solution, $\{y_n\} = \{\frac{n+1}{n}\}$, also satisfies (E_3), and (E_3) satisfies all the hypotheses of Theorem 2 except condition (7).

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