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COMMON FIXED POINT THEOREMS FOR SEQUENCES OF FUZZY MAPPINGS

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ABSTRACT. In this paper, we define g-contractive and g-contractive type fuzzy mappings and prove common fixed point theorems for sequences of fuzzy mappings on a complete metric linear space.

KEY WORDS AND PHRASES. Contractive-type fuzzy mapping, g-contractive fuzzy mapping, g-contractive type fuzzy mapping, fixed point, common fixed point.
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1. INTRODUCTION.

Fixed point theorems for fuzzy mappings were studied by Bose-Sahani, Butnariu, and others ([1]-[3]; [5]-[6]; [8]-[9]; [16]-[17]). While Weiss [17] studied a fixed point theorem for fuzzy sets, which is a fuzzy analogue of the Schauder-Tychonoff's fixed point theorem, Heilpern [9] obtained a fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of the fixed point theorems for multi-valued mappings ([7], [10], [15]) and the well-known Banach fixed point theorem. A fixed point theorem for contractive type fuzzy mappings which is a generalization of the Heilpern's result was given in [14]. In this paper, we define g-contractive and g-contractive type fuzzy mappings which are fuzzy analogues of g-contractive and g-contractive type mappings respectively ([11], [12]). For a mapping g of a complete metric linear space (X,d) into itself and a sequence $(F_i)_{i=1}^{\infty}$ of fuzzy mappings of X into $W(X)$, we consider the following conditions $(*)$ and $(**)$;

$(*)$ there exists a constant k with $0 \le k < 1$ such that for each pair of fuzzy mappings $F_i, F_j: X \to W(X), D(F_i(x), F_j(y)) \le kd(g(x), g(y))$ for all $x, y \in X$,

$(**)$ there exists a constant k with $0 \le k < 1$ such that for each pair of fuzzy mappings $F_i, F_j: X \to W(X)$ and for any $x \in X$, $\{u_x\} \subset F_i(x)$ implies that there is $\{v_y\} \subset F_j(y)$ for all $y \in X$ with $D(\{u_x\}, \{v_y\}) \le kd(g(x), g(y))$.

We show that a sequence with the condition $(*)$ satisfies the condition $(**)$, that a sequence with the condition (∗∗) has a common fixed point and consequently that a sequence with the condition (∗) has a common fixed point. These results are fuzzy analogues of common fixed point theorems for sequences of g-contractive and g-contractive type multi-valued mappings [11]. Consequently, we obtain as corollaries fixed point theorems for contractive fuzzy mappings [9] and contractive-type fuzzy mappings [14].

2. PRELIMINARIES.

We review briefly some definitions and terminologies needed ([4], [9], [16]). Let (X,d) be a metric linear space (i.e., a complex or real vector space). A fuzzy set A in X is a function with domain X and values in $[0,1]$. (In particular, if A is an ordinary (crisp) subset of X, its characteristic function χ_A is a fuzzy set with domain X and values $\{0,1\}$). Especially $\{x\}$ is a fuzzy set with membership function equal to a characterstic function of the set $\{x\}$. The α-level set of A, denoted by A_α, is defined by

$$A_\alpha = \{x: A(x) \geq \alpha\} \text{ if } \alpha \in (0,1],$$

$$A_0 = \overline{\{x: A(x) > 0\}}$$

where $\bar{B}$ denotes the closure of the (nonfuzzy) set B. $W(X)$ denotes the collection to all fuzzy sets A in X such that (i) A_α is compact and convex in X for each $\alpha \in [0,1]$ and (ii) $sup_{x \in X} A(x) = 1$. For $A, B \in W(X)$, $A \subset B$ means $A(x) \leq B(x)$ for each $x \in X$.

DEFINITION 2.1. Let $A, B \in W(X)$. Then a metric D on $W(X)$ is defined by $D(A,B) = sup_{\alpha \in [0,1]} H(A_\alpha, B_\alpha)$ where H is the Hausdorff metric in the collection $CP(X)$ of all nonempty compact subsets of X.

DEFINITION 2.2. Let X be an arbitrary set and Y be any metric linear space. F is called a fuzzy mapping iff F is a mapping from the set X into $W(Y)$.

A fuzzy mapping F is a fuzzy subset on $X \times Y$ with a membership function $F(x)(y)$. The function value $F(x)(y)$ is the grade of membership of y in $F(x)$. In case $X = Y, F(x)$ is a function from X into $[0,1]$. Especially for a multi-valued function $f: X \rightarrow 2^X, \chi_{f(x)}$ is a function from X to $\{0,1\}$. Hence a fuzzy mapping $F: X \rightarrow W(X)$ is another extension of a multi-valued function $f: X \rightarrow 2^X$.

The concept of a fuzzy set provides a natural framework for generalizing many concepts of general topology to fuzzy topology.

DEFINITION 2.3. A family $\mathfrak{T}$ of fuzzy sets in a set X is called a fuzzy topology for X and the pair $(X, \mathfrak{T})$ a fuzzy topological space, if (1) $\chi_X \in \mathfrak{T}$; (2) $\chi_\phi \in \mathfrak{T}$; (3) $\bigcup_{\lambda \in \Lambda} A_\lambda \in \mathfrak{T}$ whenever each $A_\lambda \in \mathfrak{T}$, $(\lambda \in \Lambda)$; and (4) $A \cap B \in \mathfrak{T}$ whenever $A, B \in \mathfrak{T}$. The elements of $\mathfrak{T}$ are called open and their complements closed.

If a fuzzy set A in a (crisp) topological space X satisfies $A(x) \geq lim\ sup_{n \rightarrow \infty} A(x_n)$, where $(x_n)_{n=1}^{\infty}$ is a sequence in X converging to a point $x \in X$, then A is said to be closed [17]. The fact means that the closed fuzzy set $A: X \rightarrow [0,1]$ is upper semicontinuous, i.e., a fuzzy set $1 - A$ is lower semicontinuous [13]. Thus we are led to the following definition:

DEFINITION 2.4 [17]. The induced fuzzy topology on a (crisp) topological space $(X, \mathfrak{T})$, denoted by $F(\mathfrak{T})$, is the collection of all lower semicontinuous fuzzy sets in X.

It is known that a fuzzy set A is open in a fuzzy topological space $(X, F(\mathfrak{T}))$ [respectively, closed] if and only if for each $\alpha \in [0,1], \{x \in X \mid A(x) > \alpha\}$ is open in a (crisp) topological space $(X, \mathfrak{T})$ [respectively, $\{x \in X \mid A(x) \geq \alpha\}$ is closed]. Recall that a function $F(x): X \rightarrow [0,1]$ is upper semicontinuous for each $x \in X$, where F is a fuzzy mapping defined on a metric linear space (X,d) [14].

3. COMMON FIXED POINT THEOREMS FOR SEQUENCES OF FUZZY MAPPINGS.

In this section, we introduce the notions of g-contractive and g-contractive type fuzzy mappings. We show that a sequence of fuzzy mappings with the condition (*) satisfies the condition (**), and a sequence with the condition (**) has a common fixed point. Consequently, we obtain that a g-contractive fuzzy mapping is g-contractive type, and that a g-contractive type fuzzy mapping has a fixed point.

DEFINITION 3.1. Let g be a mapping from a metric linear space (X,d) to itself. A fuzzy mapping $F: X \to W(X)$ is g-contractive if $D(F(x),F(y)) \leq kd(g(x),g(y))$ for all $x,y \in X$, for some fixed $k, 0 \leq k < 1$.

PROPOSITION 3.2 [14]. Let (X,d) be a complete metric linear space, $F: X \to W(X)$ a fuzzy mapping and $x_0 \in X$. Then there exists $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.

DEFINITION 3.3 [14]. Let (X,d) be a complete metric linear space. We call a fuzzy mapping $F: X \to W(X)$ contractive-type if for all $x \in X$, $\{u_x\} \subset F(x)$ there exists $\{v_y\} \subset F(y)$ for all $y \in X$ such that $D(\{u_x\},\{v_y\}) \leq kd(x,y)$ for some fixed k, $0 \leq k < 1$.

A metric D [respectively, Hausdorff metric H] is a metric on $W(X)$ [respectively, $CP(X)$] such that $D(\{x\},\{y\}) = d(x,y)$ [respectively, $H(\{x\},\{y\}) = d(x,y)$]. Hence D [respectively, H] is a generalization of the metric d to fuzzy sets [respectively, crisp sets].

DEFINITION 3.4. Let g be a mapping from a complete metric linear space (X,d) to itself. We call a fuzzy mapping $F: X \to W(X)$ g-contractive type if for all $x \in X$, $\{u_x\} \subset F(x)$ there exists $\{v_y\} \subset F(y)$ for all $y \in X$ such that $D(\{u_x\},\{v_y\}) \leq kd(g(x),g(y))$ for some fixed k, $0 \leq k < 1$.

We consider an example of a g-contractive type fuzzy mapping which is not contractive-type.

EXAMPLE 3.5. Let (X,d) be a Euclidean metric space $([0,\infty), |\cdot|)$. Define $F: X \to W(X)$ as follows:

$$F(x)(z) = \begin{cases} 1, & 0 \leq z \leq 2x \\ 0, & z > 2x \end{cases}$$

and define $g: [0,\infty) \to [0,\infty)$ by $g(x) = 3x$. Then F is not contractive-type but g-contractive type.

THEOREM 3.6. Let g be a mapping from a complete metric linear space (X,d) to itself. If $(F_i)_{i=1}^{\infty}$ is a sequence of fuzzy mappings of X into $W(X)$ satisfying the condition (*), then $(F_i)_{i=1}^{\infty}$ satisfies the condition (**).

PROOF. Let $x,y \in X$. If $D(F_i(x),F_j(y)) \leq kd(g(x),g(y))$ for some fixed k, $0 \leq k < 1$, then $H(F_i(x)_\alpha, F_j(y)_\alpha) \leq kd(g(x),g(y))$ for each $\alpha \in [0,1]$. Define $(f_i)_\alpha: X \to CP(X)$ by $(f_i)_\alpha(x) = F_i(x)_\alpha$ for each $\alpha \in [0,1]$, then $H((f_i)_\alpha(x),(f_j)_\alpha(y)) = H(F_i(x)_\alpha, F_j(y)_\alpha) \leq kd(g(x),g(y))$ for each $\alpha \in [0,1]$. Thus, for each $x \in X, u_x \in (f_i)_\alpha(x)$, there exists $v_y \in (f_j)_\alpha(y)$ for all $y \in X$ such that $H(\{u_x\},\{v_y\}) \leq kd(g(x),g(z))$ for each $\alpha \in [0,1]$. Since $u_x \in F_i(x)_1$ and $v_y \in F_j(y)_1, \{u_x\} \subset F_i(x)$ and $\{v_y\} \subset F_j(y)$. Hence for any $x \in X, \{u_x\} \subset F_i(x)$, there exists $\{v_y\} \subset F_j(y)$ for all $y \in X$ such that $D(\{u_x\},\{v_y\}) = H(\{u_x\},\{v_y\}) \leq kd(g(x),g(y))$ for some fixed k, $0 \leq k < 1$.

The converse of Theorem 3.6 does not hold in general.

EXAMPLE 3.7. Let g be an identity mapping from a Euclidean metric space $([0,\infty), |\cdot|)$ to itself. Let $(F_i)_{i=1}^{\infty}$ be a sequence of fuzzy mappings from $[0,\infty)$ into $W([0,\infty))$, where $F_i(x): [0,\infty) \to [0,1]$ is defined as follows:

$$\text{if } x = 0, F_i(x)(z) = \begin{cases} 1, & z = 0 \\ 0, & z \neq 0, \end{cases}$$

otherwise,
$$F_i(x)(z) = \begin{cases} 1, & 0 \le z \le \frac{x}{2} \\ \frac{1}{2}, & \frac{x}{2} < z \le ix \\ 0, & z > ix. \end{cases}$$

Then the sequence $(F_i)_{i=1}^{\infty}$ satisfies the condition (**), but does not satisfy the condition (*).

COROLLARY 3.8 [14]. Let (X,d) be a complete metric linear space. If $F: X \to W(X)$ is a contractive fuzzy mapping, then it is contractive-type.

COROLLARY 3.9. Let g be a mapping from a complete metric linear space (X,d) to itself. If $F: X \to W(X)$ is a g-contractive fuzzy mapping, then F is g-contractive type.

Weiss [17] proved a generalization to fuzzy sets of the Schauder-Tychonoff theorem by means of the classical Schauder-Tychonoff theorem, and Butnariu [2] proved that a convex and closed fuzzy mapping F defined over a nonempty convex compact subset of a real topological vector space, locally convex and Hausdorff separated, has a fixed point. Also he showed that a F-continuous fuzzy mapping defined over a nonempty convex compact subset of a n-dimensional Euclidean space $R^n (n \in N)$ has a fixed point.

Now we prove our main theorem which extends the result of Heilpern [9] on fuzzy contraction mappings and the result of Lee-Cho [14] on contractive-type fuzzy mappings to the case of a sequence of fuzzy mappings on a complete metric linear space.

THEOREM 3.10. Let g be a non-expansive mapping from a complete metric linear space (X,d) to itself. If $(F_i)_{i=1}^{\infty}$ is a sequence of fuzzy mappings of X into $W(X)$ satisfying the condition (**), then there exists $p \in X$ such that $\{p\} \subset \bigcap_{i=1}^{\infty} F_i(p)$.

PROOF. Let $x_0 \in X$. Then we can choose $x_1 \in X$ with $d(x_0, x_1) > 0$ such that $\{x_1\} \subset F_1(x_0)$ by Proposition 3.2. By the condition (**), there exists $x_2 \in X$ such that $\{x_2\} \subset F_2(x_1)$ with $D(\{x_1\},\{x_2\}) \le kd(g(x_0), g(x_1)) \le kd(x_0, x_1)$, for some fixed k, $0 \le k < 1$. Inductively, we obtain a sequence $(x_n)_{n=1}^{\infty}$ in X such that $\{x_{n+1}\} \subset F_{n+1}(x_n)$ and $D(\{x_1\},\{x_{n+1}\}) \le kd(g(x_{n-1}), g(x_n)) \le kd(x_{n-1}, x_n)$. This leads to $\{x_{n+1}\} \subset F_{n+1}(x_n)$ and $d(x_n, x_{n+1}) \le k^n d(x_0, x_1)$ for all n. Since $d(x_n, x_m) = D(\{x_n\},\{x_m\}) \le \frac{k^n}{1-k} D(\{x_0\},\{x_1\}) < \frac{k^n}{1-k} d(x_0, x_1)$ for $m > n, d(x_n, x_m) \to 0$ as $m, n \to \infty$. By the completeness of X we find an element $p \in X$ with $x_n \to p$ as $n \to \infty$. Let F_m be an arbitrary member of $(F_i)_{i=1}^{\infty}$. Since $\{x_n\} \subset F_n(x_{n-1})$ for all n, there exists $\{v_n\} \subset F_m(p)$ such that $D(\{x_n\},\{v_n\}) \le kd(g(x_{n-1}), g(p)) \le kd(x_{n-1}, p)$. But we have $d(p, v_n) \le d(p, x_n) + d(x_n, v_n) \le d(p, x_n) + kd(x_{n-1}, p)$ which implies $d(p, v_n) \to 0$ as $n \to \infty$. Since $F_m(p): X \to [0,1]$ is upper semicontinuous, $\limsup_{n \to \infty} F_m(p)(v_n) \le F_m(p)(p)$. Since $F_m(p)(v_n) = 1$ for all $n, F_m(p)(p) = 1$. Hence $\{p\} \subset F_m(p)$ for all m, that is, $\{p\} \subset \bigcap_{i=1}^{\infty} F_i(p)$.

REMARK. The sequence $(F_i)_{i=1}^{\infty}$ in Example 3.7 has a common fixed point $x = 0$.

COROLLARY 3.11. Let g be a non-expansive mapping from a complete metric linear space (X,d) to itself. If $(F_i)_{i=1}^{\infty}$ is a sequence of fuzzy mappings of X into $W(X)$ satisfying the condition (*), then there exists $p \in X$ such that $\{p\} \subset \bigcap_{i=1}^{\infty} F_i(p)$.

COROLLARY 3.12. Let g be a non-expansive mapping from a complete metric linear space (X,d) to itself. If $F: X \to W(X)$ is a g-contractive type fuzzy mapping, then there exists $p \in X$ such that $\{p\} \subset F(p)$.

COROLLARY 3.13 [14]. Let (X,d) be a complete metric linear space. If $F: X \to W(X)$ is a contractive-type fuzzy mapping, then there exists $p \in X$ such that $\{p\} \subset F(p)$.

COROLLARY 3.14 [9]. Let X be a complete metric linear space and F a fuzzy mapping from X to $W(X)$ satisfying the following condition; there exists $q \in (0,1)$ such that $D(F(x), F(y)) \leq qd(x,y)$ for each $x, y \in X$. Then there exists $p \in X$ such that $\{p\} \subset F(p)$.

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