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VARIATIONAL FINITE ELEMENT APPROACH TO A HEAT FLOW PROBLEM IN HUMAN LIMBS

V. P. SAXENA and M. P. GUPTA

School of Mathematics and Allied Sciences
Jiwaji University
Gwalior, India

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ABSTRACT. An attempt has been made to study cross sectional temperature distribution in in-vivo tissues of a human limb employing variational finite element approach. The outermost surface of the limb is assumed to be exposed to the atmosphere. The physiological and physical parameters like rate of metabolic heat generation (rmhg), blood mass flow rate (bmfr) and thermal conductivity are assumed to vary in the subregions independently. Numerical results have been obtained for various cases of practical interest.

KEY WORDS AND PHRASES: Metabolic heat generation; blood mass flow; thermal conductivity; finite element method.

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1. INTRODUCTION.

The temperature distribution in the peripheral region of limbs of a human body undergo frequent changes on account of changes in atmospheric temperature, the core being maintained at a nearly uniform temperature. The peripheral region comprises of non-uniform layers and has variable blood flow, metabolic heat generation and allied process. This intermediatory region play a very important role in maintaining a balance between the atmospheric variations and the interior consisting of intra-abdominal, intrathoracic and intracranial regions. Usually rectal and oesophageal temperatures are taken to represent core temperature. Oesophageal temperature taken at heart level is a good index of rapid changes of cardiac and aortic blood temperature. The temperature of the limbs and the surface layer of the trunk exhibits a wide variation of temperature.

This paper employs a variational finite element approach to study the temperature distribution in a normal cross-sectional region of a limb. Due to unsymmetric situations of large blood vessels passing through the core of the limb the inter-face has angular variation. The peripheral part of limb is directly exposed to atmosphere.

The peripheral part, assumed to be annular in geometry, has been approximated by the assembly of triangular elements of different sizes. Thus the circular boundaries are approximated by the polygonal one's. Different types of variations of parameters have been considered for different natural subregions such as stratum corneum, stratum germinitivum, dermis and underlying tissue (Montagana [1], Jarrett [2] and Gray [3]). Finite element formulation provides necessary flexibility in taking care of different behavior of distinctly different subregions.

2. MATHEMATICAL MODEL.

The rate of change of temperature u at a point in in vivo tissue at time t is given by the following partial differential equation (Perl [4])

$$e\bar{c}\frac{\partial u}{\partial t} = \text{div}[K \text{ grad } u] + m_b c_b (u_A - u) + S \tag{2.1}$$

where e, $\bar{c}$ and K are respectively density, specific heat and thermal conductivity of the tissue; m_b and c_b are blood mass flow rate and specific heat of the blood respectively; S is the rate of metabolic heat generation per unit volume and u_A is the arterial blood temperature. Above equation has been modified and extensively used by Saxena [5], Saxena and Arya [6], Saxena and Bindra [7,8] in the thermal study of human skin and subcutaneous tissue. Here we employ the same for a human limb with circular symmetry.

The surface of the limb is assumed to be exposed to the atmosphere at temperature u_a. The heat transfer coefficient between skin and the atmosphere may be due to convection, radiation and evaporation. Hence the boundary condition at skin surface can be put as

$$-K\frac{\partial u}{\partial n} = h(u - u_a) + LE \tag{2.2}$$

where n is the direction of the normal, h is coefficient of convection, L is the latent heat of evaporation, E is the rate of sweat evaporation. At the inner boundary we put

$$u(x,y) = f(x,y) \left| \begin{matrix} (10d \le x \le 18d) \\ (0 \le y \le 4d) \end{matrix} \right. \tag{2.3}$$

$$\frac{\partial u}{\partial y} = 0 \qquad \left| \begin{matrix} (0 \le x \le 10d) \\ (18d \le x \le 28d) \end{matrix} \right. \tag{2.4}$$

where d is a distance constant. The boundary condition (2.4) corresponds to the case of mirror symmetry in temperature distribution about x-axis (horizontal diameter). The variational form of equation (2.1) for a two dimensional steady state case along with the boundary condition (2.2) is given by (Myers [9])

$$I = \frac{1}{2}\int\int_A \left[K\left(\frac{\partial u}{\partial x}\right)^2 + K\left(\frac{\partial u}{\partial y}\right)^2 + m_b c_b (u_A - u)^2 - 2Su\right] dx\, dy + \int_\Omega [h(u - u_a)^2 + 2LEu] d\Omega \tag{2.5}$$

Here the problem region A with boundary Ω is a cylindrical limb with circular cross-sectional and symmetrical with respect to x-axis. The region of interest is semi-circular and is descretized into 150 triangular elements and 96 nodes as shown in Fig. 1. Here the angular points of each element are the nodal points.

The integral in equation (2.5) may be written as

$$I = I_k + I_m + I_s + I_\Omega \tag{2.6}$$

where

$$m = m_b c_b$$

$$I_k = \frac{1}{2}\int\int_A K\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dxdy \tag{2.7}$$

$$I_m = \frac{1}{2}\int\int_A m(u_A - u)^2 dxdy \tag{2.8}$$

$$I_s = -\int\int_A Su\, dxdy \tag{2.9}$$

$$I_\Omega = \frac{1}{2}\int_\Omega [h(u - u_a)^2 + 2LEu]d\,\Omega \tag{2.10}$$

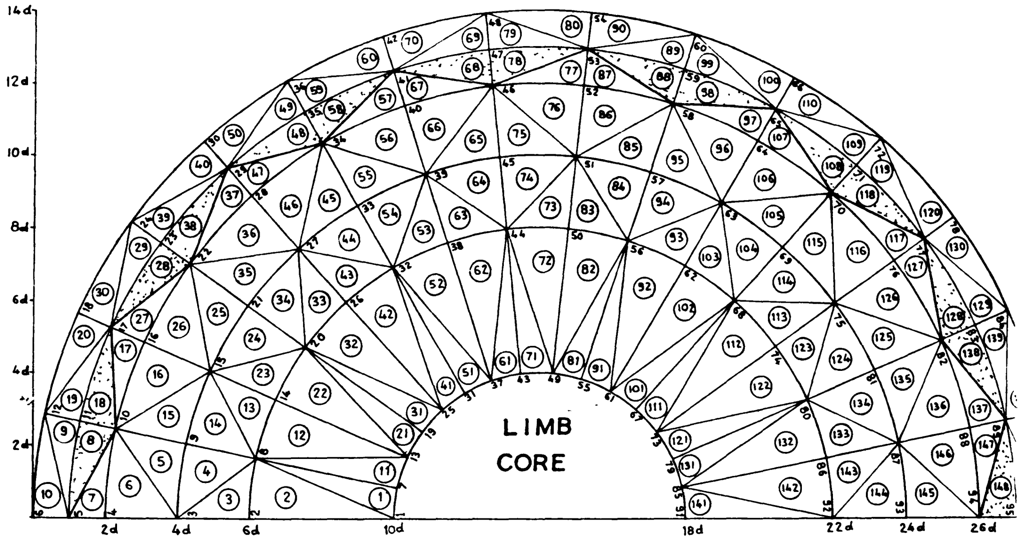

Fig. 1. Triangular Element Arrangement for Annular Cross Section of Human Limbs Numerals Without Circles Nodes and with Circles Denote Element Numbers

Next we extremize I by differentiating it with respect to each nodal temperature u_i and setting derivatives equal to zero. That is

$$\frac{dI}{dU} = \frac{dI_r}{dU} = 0, \quad r = k, m, s, \Omega \tag{2.11}$$

where

$$U = \begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_i \\ u_j \\ u_k \\ \cdot \\ \cdot \\ \cdot \\ u_{96} \end{bmatrix}$$

In view of the biology the region under study has been divided into one hundred and fifty triangular elements so that the integrals in equation (2.11) are computed as a sum over each of the elements. Hence

$$\frac{dI_r}{dU} = \sum_{e=1}^{150} \frac{dI_r^{(e)}}{dU}, \quad r = k, m, s, \Omega . \tag{2.13}$$

Here (e) stands for the elements whose nodes are i, j and k. The expression on the right-hand side of equation (2.13) for $r = \Omega$ will be summed up for the elements on the boundary of the outer surface of the limb. Equations (2.13) will be commuted separately and then substituted into equation (2.11). For the (e)th element u_i, u_j and u_k are the only temperatures to be taken into account. For this element $I_k^{(e)}$, $I_m^{(e)}$ and $I_s^{(e)}$ will be function of these three corner temperatures only. Whereas $I_\Omega^{(e)}$ will be function of only two corner temperatures which lie on the outer boundary and element (e) adjoining this boundary. Consequently, the partial derivatives of $I_r^{(e)}$ for $r = k, m, s$, with respect to all other nodal temperatures will be zero.

3. SOLUTION.

The following linear variation of temperature within each element is expressed as

$$u^{(e)} = p^T C^{(e)} \tag{3.1}$$

where

$$p^T = [1, x, y], \quad C^{(e)} = \begin{bmatrix} c_1^{(e)} \\ c_2^{(e)} \\ c_3^{(e)} \end{bmatrix}$$

Now $u^{(e)}$ is equal to u_i, u_j and u_k at the corners of the eth element whose nodal temperatures are u_i, u_j and u_k. Thus we have

$$U^{(e)} = p^{(e)} C^{(e)} \tag{3.2}$$

where

$$U^{(e)} = \begin{bmatrix} u_j \\ u_j \\ u_k \end{bmatrix}, \quad p^{(e)} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}$$

Now solving equation (3.2) for $C^{(e)}$ as given below

$$C^{(e)} = R^{(e)} U^{(e)} \tag{3.3}$$

where

$$R^{(e)} = p^{(e)^{-1}}$$

Substituting the value of $C^{(e)}$ from equation (3.3) in equation (3.1) we obtain

$$u^{(e)} = p^T R^{(e)} U^{(e)} \tag{3.4}$$

With the help of equation (3.4), we can evaluate the integral I. Further differentiating I with respect to each nodal temperatures and setting derivatives equal to zero we arrive at the following system of simultaneous equations

$$WU = Z \tag{3.5}$$

where W is a matrix of order (96 × 96) and Z is a matrix of order (96 × 1). Finally solving the system of simultaneous equations (3.5) we obtain the values of nodal temperatures (see Table I).

4. NUMERICAL RESULTS

Assumptions regarding thermal conductivity, metabolic heat generation and blood mass flow rate are given as under (Knudsen and Overgaard [10], Saxena and Bindra [7,8]).

For the Elements of Subregions	$K^{(e)}$ cal/ cm-min-degc	$S^{(e)}$	$M^{(e)}$
Ist	0.060	$S_{max} = s$	$M_{max} = m$
IInd	0.050	$7/8s$	$2/3m$
IIIrd	0.040	$5/8s$	$1/3m$
IVth	0.035	$3/8s$	$1/6m$
Vth	0.030	$1/8s$	0.0
VIth	0.030	0.0	0.0

The numerical results have been computed for two different cases of atmospheric temperatures as given below

	Case-I	Case-II
	$u_a = 15°C$	$u_a = 23°C$
m cal./cm^3-min.-deg C	0.003	0.018
s cal./cm^3-min.	0.0357	0.018
E gm/cm^3-min.	0.0	0.12×10^{-3} 0.24×10^{-3}

$L = 579$ cal/gm, $h = 0.009$ cal/cm-min-deg.C.

The constant 'd' can be assigned any value depending on the sample of the limb under study. Here we have taken

$$d = 0.625\ cm$$

The values of nodal temperatures have been obtained for two cases of atmospheric temperatures and are given in Table I. The temperatures at the nodes with numbers 1,7,13,19,25,31,37,43,49,55,67,73,79,85 and 91 (along the inner boundary) reflect the variation in the limb core temperature with respect to position.

The effect of this variation in limb core temperature on the other subregions is visible in Table I. On comparing the nodal temperature on outer surface and in each subregion, it is observed that these nodal temperatures vary considerably with the change in atmospheric temperatures and rate of sweat evaporation.

The finite element method used here has made it possible to include more details of biology such as the wavy structure of the layers and different values of physical and physiological parameters in each subregion. A computer program was developed and executed on ICIM 6040 to perform these computations.

Table I. Values of Nodal Temperatures for Different Cases of Atmospheric Temperatures

Nodal	$u_a = 15°C$	$u_a = 23°C$	
Temperatures	$E = 0$	$E = 0.12 \times 10^{-3}\Omega$	$E = 0.24 \times 10^{-3}\Omega$
u1	30.00	30.00	30.00
u2	29.51	29.91	29.22
u3	29.06	28.85	27.56
u4	26.63	26.65	24.27
u5	25.14	25.21	22.14
u6	23.49	23.60	19.77
u7	30.26	30.26	30.26
u8	29.82	30.01	29.32
u9	29.02	28.84	27.51
u10	26.74	26.78	24.43
u11	25.08	25.16	22.04
u12	23.42	23.53	19.65
u13	30.53	30.53	30.53
u14	30.47	30.12	29.42
u15	29.18	29.05	27.76
u16	26.69	26.78	24.38
u17	25.21	25.34	22.26
u18	23.57	23.73	19.91
u19	30.80	30.80	30.80
u20	30.68	30.36	29.68
u21	29.24	29.16	27.83
u22	26.89	27.03	24.68
u23	25.19	25.35	22.22
u24	23.54	23.72	19.84
u25	31.06	31.06	31.06
u26	30.79	30.54	29.84
u27	29.44	29.43	28.14
u28	26.89	27.08	24.68
u29	25.40	25.62	22.55
u30	23.70	23.93	20.09
u31	31.33	31.33	31.33
u32	31.03	30.81	30.13

Nodal Temperatures	$u_a = 15°C$	$u_a = 23°C$	
	$E = 0$	$E = 0.12 \times 10^{-3}\Omega$	$E = 0.24 \times 10^{-3}\Omega$
u33	29.51	29.55	28.21
u34	27.13	27.38	25.03
u35	25.39	25.64	22.51
u36	23.70	23.95	20.08
u37	31.60	31.60	31.60
u38	31.16	31.00	30.31
u39	29.76	29.86	28.56
u40	27.15	27.45	25.06
u41	25.59	25.90	22.83
u42	23.88	24.19	20.36
u43	31.86	31.86	31.86
u44	31.43	31.30	30.63
u45	29.84	29.90	28.66
u46	27.38	27.73	25.37
u47	25.65	25.99	22.88
u48	23.88	24.21	20.34
u49	32.13	32.13	32.13
u50	31.55	31.48	30.79
u51	30.07	30.28	28.98
u52	27.41	27.81	25.41
u53	25.80	26.20	23.11
u54	24.07	24.46	20.63
u55	32.40	32.40	32.40
u56	31.79	31.76	31.08
u57	30.17	30.41	29.08
u58	27.65	28.10	25.74
u59	25.87	26.29	23.18
u60	24.07	24.46	20.59
u61	32.66	32.66	32.66
u62	31.92	31.94	31.25
u63	30.41	30.73	29.43
u64	27.68	28.17	25.78
u65	26.07	26.55	23.49
u66	24.26	24.71	20.89
u67	32.93	32.93	32.93
u68	32.16	32.22	31.54
u69	30.47	30.83	29.50
u70	27.93	28.47	26.13
u71	26.04	26.54	23.40
u72	24.23	24.70	20.81
u73	33.20	33.20	33.20
u74	32.26	32.38	31.69

Nodal Temperatures	$u_a = 15°C$	$u_a = 23°C$	
	$E = 0$	$E = 0.12 \times 10^{-3}\Omega$	$E = 0.24 \times 10^{-3}\Omega$
u75	30.67	31.11	29.80
u76	27.89	28.49	26.09
u77	26.31	26.88	23.83
u78	24.38	24.90	21.06
u79	33.46	33.46	33.46
u80	32.48	32.64	31.96
u81	30.73	31.20	29.86
u82	28.12	28.77	26.43
u83	26.18	26.77	23.62
u84	24.39	24.93	21.06
u85	33.73	33.73	33.73
u86	32.52	32.73	32.04
u87	30.88	31.42	30.12
u88	28.07	28.77	26.38
u89	26.38	27.04	23.96
u90	24.54	25.14	21.31
u91	34.00	34.00	34.00
u92	32.64	32.87	32.19
u93	30.85	31.38	30.05
u94	28.22	28.94	26.59
u95	26.29	26.94	23.81
u96	24.46	25.05	21.18

where $\Omega = gm/cm^3 - \text{min}$

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