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GLIDING HUMP PROPERTIES AND SOME APPLICATIONS

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ABSTRACT. In this note we consider several types of gliding hump properties for a sequence space E and we consider the various implications between these properties. By means of examples we show that most of the implications are strict and they afford a sort of structure between solid sequence spaces and those with weakly sequentially complete β-duals. Our main result is used to extend a result of Bennett and Kalton which characterizes the class of sequence spaces E with the property that $E \subset S_F$ whenever F is a separable FK space containing E where S_F denotes the sequences in F having sectional convergence. This, in turn, is used to identify a gliding humps property as a sufficient condition for E to be in this class.

KEY WORDS AND PHRASES. Gliding hump properties, weak sequential completeness of the β-dual, sectional convergence in FK spaces, Theorem of Schur, Theorem of Hahn.
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1. INTRODUCTION.

Over the past eighty years the "gliding hump" technique has been a frequently used tool to establish results in summability and sequence space theory. Among the more familar examples would be the Silverman–Toeplitz theorem which gives necessary and sufficient conditions for the regularity of a summability method [22], the Mazur–Orlicz bounded consistency theorem ([6], [12] and [13]), the theorem of Köthe and Toeplitz on the weak sequential completeness of the Köthe dual of a solid sequence space [11] and the theorems of Schur on the characterization of coercive matrices and the equivalence of weak and strong convergence in ℓ_1 [19]. Whereas the first three of these have subsequently been argued using functional analytic techniques (see e.g. [24] and [10]) no such "soft" proofs of Schur's theorems are known. Various authors have considered sequence spaces enjoying certain gliding hump type properties. See for example, [8] for extensions of Schur's theorems, ([4], [5], [20]) for Mazur–Orlicz type theorems and ([5], [14]) for weak sequential completeness results. The gliding hump technique has also proven to be a key ingredient in the solution to problems related to the Wilansky Property ([1], [21], [15]).

In section 3 of this note we introduce various types of gliding hump properties and discuss the implications between them. We give examples in section 5 to show that most of these implications are strict and they are, in some sense, affording a structure to the set of sequence spaces between the solid spaces and those with weakly sequentially complete β-duals.

In [2, Theorem 6], Bennett and Kalton characterized the class of sequence spaces E for which $E \subset S_F$ whenever F is a separable FK space containing E (here, S_F denotes the elements of F having sectional convergence). In Theorem 3.6 we extend their result by showing that it suffices to consider only the case where F is a convergence domain of a matrix. Combining this observation with our main result Theorem 3.5 we obtain in corollary 3.7 the more tractable pointwise weak gliding hump property (see definiton 3.1 below) as a sufficient condition for E to belong to this class. In section 4 we apply the techniques of this paper to obtain short proofs of some classical results.

2. NOTATION AND PRELIMINARIES.

Let ω denote the linear space of all scalar (real or complex) sequences. By a sequence space E we shall mean any linear subspace of ω. A sequence space E endowed with a locally convex topology is called a K-space if the inclusion map $\imath : E \longrightarrow \omega$ is continuous where ω has the topology of coordinatewise convergence. A K-space E with a Fréchet topology is called an FK-space. If, in addition, the topology is normable then E is called a BK-space. We assume throughout this note familarity with the standard sequence spaces and their natural topologies (see e. g. [24], [9]).

For a sequence space E the multiplier space of E and the β-dual of E are given by

$$\mathcal{M}(E) = \left\{x \in \omega \;\middle|\; xy \in E \text{ for each } y \in E\right\}$$

and

$$E^{\beta} = \left\{x \in \omega \;\middle|\; \sum_k x_k y_k \text{ converges for each } y \in E\right\}$$

where xy denotes the coordinatewise product. For $x \in \omega$, $n \in \mathbb{N}$ the n^{th} section of x is

$$x^{[n]} = \sum_{k=1}^{n} x_k e^k$$

where $e^k = (\delta_{ik})_{i=1}^{\infty}$ is the k^{th} coordinate vector. For any positive term sequence $\mu = (\mu_k)$ let

$$E_{\mu} = \left\{x \in \omega \;\middle|\; \left(\frac{x_k}{\mu_k}\right) \in E\right\}.$$

If (E, F) is a dual pair then $\sigma(E, F)$, $\tau(E, F)$ denotes the weak topology and the Mackey topology respectively. For a sequence space E and a linear subspace F of E^{β}, (E, F) is a dual pair under the natural bilinear form

$$\langle x, y\rangle = \sum_k x_k y_k .$$

If E is a K-space containing φ, the space of finitely non-zero sequences, we let

$$\begin{aligned} L_E &= \left\{x \in E \;\middle|\; \{x^{[n]} | n \in \mathbb{N}\} \text{ is bounded in } E\right\} \\ W_E &= \left\{x \in E \;\middle|\; x^{[n]} \longrightarrow x \quad \sigma(E, E')\right\} \\ S_E &= \left\{x \in E \;\middle|\; x^{[n]} \longrightarrow x \text{ in } E\right\} \end{aligned}$$

where E' denotes the topological dual of E. A K-space E containing φ with $E = S_E$ is called an AK-space.

If $A = (a_{nk})$ is an infinite matrix with scalar entries the convergence domain

$$c_A = \left\{x \in \omega \;\middle|\; Ax = \left(\sum_k a_{nk} x_k\right)_{n=1}^{\infty} \in c\right\}$$

admits a natural FK topology [21]. For $x \in c_A$ we write $\lim_A x = \lim Ax$.

If $\varphi \subset c_A$ let $a_k = \lim_n a_{nk}$ and define

$$l_A = \left\{ x \in c_A \mid \sum_k a_k x_k \text{ exists} \right\},$$

$\Lambda_A : l_A \longrightarrow \mathbb{K}$ by $\Lambda_A(x) = \lim_A x - \sum_k a_k x_k$ (where $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$) and

$$\Lambda_A^\perp = \left\{ x \in l_A \mid \Lambda_A(x) = 0 \right\}.$$

Further if $\varphi \subset c_A$ we write L_A, W_A, S_A instead of $L_{c_A}, W_{c_A}, S_{c_A}$. In this case $W_A = L_A \cap \Lambda_A^\perp$ (see e. g. [24]).

3. THE GLIDING HUMP PROPERTIES.

We begin by introducing several types of gliding hump properties.

DEFINITION 3.1 . A sequence $(y^{(n)})$ in $\omega \setminus \{0\}$ is called a block sequence if there exists an index sequence (k_j) such that $y_k^{(n)} = 0$ for any $n, k \in \mathbb{N}$ with $k \notin]k_{n-1}, k_n]$ where $k_0 := 0$, and it is called a 1-block sequence if furthermore $y_k^{(n)} = 1$ for each $k \in]k_{n-1}, k_n]$ and $n \in \mathbb{N}$.

Let E be a sequence space containing φ.

- E has the gliding hump property (ghp) if for each block sequence $(y^{(n)})$ satisfying $\sup_{n\in\mathbb{N}} \|y^{(n)}\|_{bv} < \infty$ and any monotonicly increasing sequence (n_k) of integers there exists a subsequence (m_k) of (n_k) with $\sum_{j=1}^{\infty} y^{(m_j)} \in E$ (pointwise sum).
- E has the pointwise gliding hump property (p_ghp) if for each $x \in E$, any block sequence $(y^{(n)})$ satisfying $\sup_{n\in\mathbb{N}} \|y^{(n)}\|_{bv} < \infty$ and any monotonicly increasing sequence (n_k) of integers there exists a subsequence (m_k) of (n_k) with $\sum_{j=1}^{\infty} xy^{(m_j)} \in E$ (pointwise sum).
- E has the uniform gliding hump property (u_ghp) if the sequence (m_k) in the definition of the p_ghp may be chosen independently of $x \in E$.
- E has the pointwise weak gliding hump property (p_wghp) if the definition of the p_ghp is fulfilled for each 1-block sequence.
- E has the uniform weak gliding hump property (u_wghp) if the definition of the u_ghp is fulfilled for each 1-block sequence.

We say that E has the strong p_ghp (u_ghp, p_wghp or u_wghp) if $\sum_{j=1}^{\infty} xy^{(m_j)} \in E$ (pointwise sum) holds for any subsequence of (m_k) in the above definitions; in this case, we use the notation sp_ghp, su_ghp, sp_wghp and su_wghp, respectively.

REMARKS 3.2 . Let E be a sequence space containing φ.

(a) Obviously, the definition of the ghp corresponds with the definition given in [20],[4] and the definition of the p_wghp corresponds to the weak gliding hump property considered by D. Noll [14].

(b) E has the u_ghp if and only if $\mathcal{M}(E)$ has the ghp.

(c) su_ghp $\Longrightarrow$ su_wghp $\Longrightarrow$ u_wghp $\Longrightarrow$ p_wghp;
su_ghp $\Longrightarrow$ sp_ghp $\Longrightarrow$ sp_wghp $\Longrightarrow$ p_wghp;
su_ghp $\Longrightarrow$ u_ghp $\Longrightarrow$ p_ghp $\Longrightarrow$ p_wghp;

su_wghp $\Longrightarrow$ sp_wghp and sp_ghp $\Longrightarrow$ p_ghp.

(In the last section we provide examples to show that most of these implications are strict.)

(d) Each solid space has the su_ghp and each monotone space has the su_wghp. (Note, each solid sequence space is monotone.)

(e) Examples of spaces E such that $\mathcal{M}(E)$ has the ghp may be found in [1, Remark 1].

(f) In [4] T. Leiger and the first author proved the validity of theorems of Mazur Orlicz type under the assumption that M is a sequence space such that $\mathcal{M}(M)$ has the ghp, that is, M has the u_ghp. Actually, in each instance only the fact that $\mathcal{M}(M)$ has the p_ghp was used in the arguments.

THEOREM 3.3 . Let E be an FK space containing φ. Then S_E has the strong p_ghp; in particular, if E is an FK-AK space then E has the strong p_ghp.

PROOF. The FK topology of E may be generated by seminorms

$$p_r \ (r \in \mathbb{N}) \text{ such that } p_r(x) \leq p_{r+1}(x) \quad (r \in \mathbb{N} \text{ and } x \in E). \tag{$\circ$}$$

Since S_E is an FK-AK space we may assume that E is an FK-AK-space.
Now, let $x \in E$ be given. Then

$$\sup_{\nu \geq n} p_r\left(\sum_{k=n}^{\nu} x_k e^k\right) \longrightarrow 0 \quad (n \to \infty \text{ and } r \in \mathbb{N}). \tag{$*$}$$

Further let $(y^{(j)})$ be (a subsequence of) any block sequence satisfying $M := \sup_{j\in\mathbb{N}} \|y^{(j)}\|_{bv} < \infty$. There exist index sequences (ν_j) and (μ_j) such that $\nu_j \leq \mu_j < \nu_{j+1} \quad (j \in \mathbb{N})$ and

$$y^{(j)} = \sum_{k=\nu_j}^{\mu_j} y_k^{(j)} e^k, \quad \text{thus } y_k^{(j)} = 0 \text{ for } k \notin [\nu_j, \mu_j]\,.$$

On account of ($\circ$) it is sufficient to prove $xy^{(j)} \longrightarrow 0$ in E. For that end let $r \in \mathbb{N}$ be given. Then we have

$$\begin{aligned} p_r(xy^{(j)}) = p_r\left(\sum_{k=\nu_j}^{\mu_j} x_k y_k^{(j)} e^k\right) &\leq \sup_{K \geq \nu_j} p_r\left(\sum_{k=\nu_j}^{K} x_k e^k\right) \sum_{k=\nu_j}^{\mu_j} \left|y_k^{(j)} - y_{k+1}^{(j)}\right| \\ &\leq M \sup_{K \geq \nu_j} p_r\left(\sum_{k=\nu_j}^{K} x_k e^k\right) \overset{j\to\infty}{\longrightarrow} 0 \end{aligned}$$

by ($*$) which proves $xy^{(j)} \longrightarrow 0$ in E. □

REMARK 3.4 . In general, W_E fails the p_wghp. [Example: Let Σ be the summation matrix and $E := c_{\Sigma^{-1}}$. Then W_E fails the p_wghp since $x := e = \sum_j e^j \in W_E$ (pointwise sum) and $(n_j) = (2j)$ does not have any subsequence (m_k) such that $\tilde{x} := \sum_k e^{m_k}$ (pointwise sum) $\in E$ since $\Sigma^{-1}\tilde{x} \in m_0 \setminus c$.]

THEOREM 3.5 . Let E be a sequence space containing φ, and let B be a matrix such that $E \subset c_B$. Then $E \subset S_B$ if E has the p_wghp.

PROOF. Suppose E has the p_wghp. We know from Theorem 6 of D. Noll [14] and Remark 3.2(a) that $(E^\beta, \sigma(E^\beta, E))$ is weakly sequentially complete. Therefore, by an inclusion theorem of G. Bennett and N. J. Kalton [2, Theorem 5] we get $E \subset W_B$, in particular $E \subset L_B$ and $E \subset \Lambda_B^{\frac{1}{r}}$.
Now, assume $E \subset W_B$ and $E \not\subset S_B$, that is, there exists an $x \in E \subset W_B = L_B \cap \Lambda_B^{\frac{1}{r}}$ with $x \notin S_B$, thus

$$\lim\nolimits_B x = \sum_{k=1}^{\infty} b_k x_k\,, \qquad \sup_{n,r\in\mathbb{N}} \left|\sum_{k=1}^{r} b_{nk} x_k\right| < \infty \quad \text{and} \quad \sup_{\substack{n,r\in\mathbb{N}\\ r\geq\nu}} \left|\sum_{k=\nu}^{r} b_{nk} x_k\right| \not\longrightarrow 0\,.$$

Therefore we may choose an $\eta > 0$ and index sequences (α_j), (β_j) and (n_j) with $\alpha_j \leq \beta_j$ such that

$$\Big| \sum_{k=\alpha_j}^{\beta_j} b_{n_j k} x_k \Big| \geq \eta \qquad \text{for each } j \in \mathbb{N}\,.$$

Now we employ a gliding hump argument. Let $k_0 := 1$ and choose n_1^* such that

$$\sum_{k=1}^{k_0} \Big| b_{nk} - b_k \Big| |x_k| < 2^{-1} \quad (n \geq n_1^*)\,.$$

Then there exist a $j_1 \in \mathbb{N}$ with $n_{j_1} > n_1^*$ and a $k_1 > \beta_{j_1}$ such that (note $x \in l_B$)

$$\max_{\substack{K \geq k_1 \\ p \in \mathbb{N}_0}} \left\{ \Big| \sum_{k=K}^{K+p} b_{nk} x_k \Big| , \Big| \sum_{k=K}^{K+p} b_k x_k \Big| \right\} < 2^{-1} \qquad (n \leq n_{j_1})\,.$$

Choose $n_2^* \geq n_{j_1}$ such that

$$\sum_{k=1}^{k_1} \Big| b_{nk} - b_k \Big| |x_k| < 2^{-2} \qquad (n \geq n_2^*)$$

and a $k_2 > k_1$ such that

$$\max_{\substack{K \geq k_2 \\ p \in \mathbb{N}_0}} \left\{ \Big| \sum_{k=K}^{K+p} b_{nk} x_k \Big| , \Big| \sum_{k=K}^{K+p} b_k x_k \Big| \right\} < 2^{-2} \qquad (n \leq n_2^*)\,.$$

Proceeding inductively, we get index sequences (k_ν), (j_ν), (n_ν) with

$$n_1^* < n_{j_1} < n_2^* < n_3^* < n_{j_2} < \ldots < n_{2\nu-1}^* < n_{j_\nu} < n_{2\nu}^* < n_{2\nu+1}^* < \ldots$$

and

$$k_0 < \alpha_{j_1} \leq \beta_{j_1} < k_1 < k_2 < n_{j_1} < \ldots < k_{2\nu-2} < \alpha_{j_\nu} \leq \beta_{j_\nu} < k_{2\nu-1} < k_{2\nu} < \ldots$$

fulfilling

$$\sum_{k=1}^{k_{\nu-1}} \Big| b_{nk} - b_k \Big| |x_k| < 2^{-\nu} \qquad (n \geq n_\nu^*)\,.$$

and

$$\max_{\substack{K \geq k_\nu \\ p \in \mathbb{N}_0}} \left\{ \Big| \sum_{k=K}^{K+p} b_{nk} x_k \Big| , \Big| \sum_{k=K}^{K+p} b_k x_k \Big| \right\} < 2^{-\nu} \qquad \begin{cases} \text{if } \nu \text{ is odd and } n \leq n_{j_\nu} \\ \text{if } \nu \text{ is even and } n \leq n_\nu^* \end{cases}.$$

Now, we define a subsequence $(y^{(\nu)})$ of a 1-block sequence by

$$y_k^{(\nu)} := \begin{cases} 1 & \text{if } \alpha_{j_\nu} \leq k \leq \beta_{j_\nu} \\ 0 & \text{otherwise} \end{cases}$$

and consider

$$yx \quad \text{where} \quad y := \sum_{\nu=1}^{\infty} y^{(\nu)} \text{ (pointwise sum)}\,.$$

Since E has the p_wghp we may assume that $yx \in E$ (otherwise we switch over to a subsequence $(y^{(n_k)})$ and adapt the chosen index sequences). For a proof of Theorem 3.5 it is sufficient to prove $yx =: z \notin c_B$. For this let $\nu \geq 2$ and $n := n_{j_\nu}$. Then (note, $\sum_k b_k z_k$ exists)

$$\begin{aligned}
&\Big| \sum_{k=1}^{\infty} b_{nk} z_k - \sum_{k=1}^{\infty} b_k z_k \Big| \\
&\geq \; - \sum_{k=1}^{k_{2\nu-2}} \Big| b_{nk} - b_k \Big| |x_k| + \Big| \sum_{k=\alpha_{j_\nu}}^{\beta_{j_\nu}} b_{nk} x_k \Big| - \sum_{r=\nu+1}^{\infty} \Big| \sum_{k=\alpha_{j_r}}^{\beta_{j_r}} b_{nk} x_k \Big| \\
&\geq \; \eta - 2^{-\nu} - 2^{-\nu} \longrightarrow \eta > 0 \text{ for } \nu \to \infty\,.
\end{aligned}$$

Now let $\nu \geq 2$ and $n := n^*_{2\nu}$. Then (see above)

$$\Big|\sum_{k=1}^{\infty} b_{nk} z_k - \sum_{k=1}^{\infty} b_k z_k\Big| \leq \sum_{k=1}^{\infty} \big|b_{nk} - b_k\big|\,|x_k| + \sum_{r=\nu+1}^{\infty} \Big|\sum_{k=\alpha_{1r}}^{\beta_{1r}} b_{nk} x_k\Big|$$

$$\leq 2^{-\nu} + 2^{-\nu} \longrightarrow 0 \ (\nu \to \infty).$$

Altogether we have proved $yx \notin c_B$. □

Now, an obvious question is whether the statement in Theorem 3.5 remains true if we replace the domain c_B by any separable FK space F with $E \subset F$. A positive answer is a consequence of the following theorem.

THEOREM 3.6. Let E be a sequence space containing φ. Then the following statements are equivalent:

(i) $(E, \tau(E, E^\beta))$ is an AK-space and E^β is $\sigma(E^\beta, E)$ sequentially complete.

(ii) If F is any separable FK-space with $E \subset F$ then $E \subset S_F$.

(iii) If A is any matrix with $E \subset c_A$ then $E \subset S_A$.

PROOF. The equivalence (i)$\Leftrightarrow$(ii) is Theorem 6, (i)$\Leftrightarrow$(ii) of G. Bennett and N. J. Kalton [2]. The implication (ii)$\Rightarrow$(iii) is obviously valid since domains c_A are separable FK-spaces.
We are going to prove (iii)$\Rightarrow$(i). Let (iii) be valid. Then E^β is $\sigma(E^\beta, E)$-sequentially complete by [2, Theorem 5, (iv)$\Rightarrow$(i)].
Assume, $(E, \tau(E, E^\beta))$ is not AK. Thus, we may choose an $x \in E$ and an absolutely convex $\sigma(E^\beta, E)$-compact subset K of E^β such that

$$p_K(x^{[n]} - x) \not\to 0 \ (n \to \infty) \quad \text{where} \quad p_K(z) := \sup_{a \in K} \Big|\sum_k a_k z_k\Big| \qquad (z \in E).$$

Therefore we may choose an index sequence (n_i) and a sequence $(a^{(i)})$ in K such that

$$\Big|\sum_{k=n_i+1}^{\infty} a_k^{(i)} x_k\Big| \geq \eta > 0 \qquad (i \in \mathbb{N}). \tag{$*$}$$

Since K is $\sigma(E^\beta, E)$-compact, $\sigma(E^\beta, E)$ and $\sigma(E^\beta, \varphi)$ coincide on K and $\sigma(E^\beta, \varphi)$ is metrizable we may assume that $(a^{(i)})$ is $\sigma(E^\beta, E)$-convergent to an $a \in K$. (Otherwise we switch over to a subsequence of $(a^{(j)})$.) If A denotes the matrix given by

$$a_{ik} := a_k^{(i)} \qquad (i, k \in \mathbb{N})$$

then –in summability language– the last assumption tells us

$$E \subset c_A \quad (\text{even } E \subset \Lambda_A^{\perp}).$$

From $(*)$ we get $x \notin S_A$ which contradicts the assumption that (iii) is true. □

COROLLARY 3.7. Let E be a sequence space containing φ and F be a separable FK-space with $E \subset F$. If E has the p_wghp then $E \subset S_F$.

PROOF. Theorem 3.6 and 3.5. □

COROLLARY 3.8. Let Y be a sequence space and E be an FK-space with $\varphi \subset Y \cap E$ and B be a matrix with $Y \cap S_E \subset c_B$. Then $Y \cap S_E \subset S_B$ if Y has the p_wghp.
The statement remains true if we replace c_B by any separable FK-space F.

PROOF. Corollary 3.7 and the fact that $Y \cap S_E$ has the p_wghp. □

COROLLARY 3.9 . Let E be a separable FK space containing φ such that $S_E \subsetneqq W_E$. Then W_E fails the p_wghp (whereas S_E has the strong p_ghp).

PROOF. Theorem 3.3 and Corollary 3.7. □

4. APPLICATIONS.

REMARK 4.1 . Let $A = (a_{nk})$ be a matrix with $\varphi \subset c_A$ and let $x \in c_A$. Then

$$x \in S_A \iff \sum_{k=1}^{\infty} a_{nk} x_k \text{ converges uniformly in } n \in \mathbb{N} .$$

This observation gives us a short proof of the following theorem containing a Toeplitz-Silverman theorem.

THEOREM 4.2 (matrices being conservative for $\mathbf{c_0}$). For matrices $A = (a_{nk})$ the following statements are equivalent:

(a) $c_0 \subset c_A$.

(b) $c_0 \subset S_A$.

(c) $\varphi \subset c_A$ and $\|A\| := \sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| < \infty$.

PROOF. The implication (a)$\Rightarrow$(b) comes from the AK-property of c_0 and the monotonicity of FK-topologies. (This statement follows also by Theorem 3.5 since c_0 obviously has the p_wghp.) Using standard estimations we may prove (c)$\Rightarrow$(a). We are going to prove the essential part (b)$\Rightarrow$(c).
Let $c_0 \subset S_A$. Therefore, we can apply the above remark to any $x \in c_0$.
If $\|A\| = \infty$ we may choose a sequence (n_j) in $\mathbb{N}$ and index sequences (α_j) and (β_j) with $\alpha_j \le \beta_j < \alpha_{j+1}$ ($j \in \mathbb{N}$) such that

$$\sum_{k=\alpha_j}^{\beta_j} |a_{n_jk}| \ge j^2 \qquad (j \in \mathbb{N}) .$$

Defining $y \in c_0$ by

$$y_k := \begin{cases} \frac{1}{j} \operatorname{sgn} a_{n_jk} & \text{if } \alpha_j \le k \le \beta_j \\ 0 & \text{otherwise} \end{cases}$$

we get

$$\Big| \sum_{k=\alpha_j}^{\beta_j} a_{n_jk} y_k \Big| = \frac{1}{j} \sum_{k=\alpha_j}^{\beta_j} |a_{n_jk}| \ge j \qquad (j \in \mathbb{N}) .$$

Thus $\sum_{k=1}^{\infty} a_{nk} y_k$ does not converge uniformly in $n \in \mathbb{N}$ which contradicts $c_0 \subset S_A$. □

Using the same method we get also a proof of a theorem containing a theorem of Hahn (equivalence of (a) and (c)). However, we should mention that the proof of '(a)$\Longrightarrow$(c)' presented in [18, Theorem 4.1, p. 110] is more elegant.

THEOREM 4.3 (matrices summing each absolute summable sequence) . For matrices $A = (a_{nk})$ the following statements are equivalent:

(a) $\ell \subset c_A$.

(b) $\ell \subset S_A$.

(c) $\varphi \subset c_A$ and $\sup_{n,k \in \mathbb{N}} |a_{nk}| < \infty$.

PROOF. (a)$\Rightarrow$(b) follows from the continuity of the inclusion map and the fact that ℓ is an FK–AK-space and the monotonicity of FK-topologies whereas (c)$\Rightarrow$(a) may be proved with classical estimations. '(b)$\Rightarrow$(c)': Let $\ell \subset S_A$. Thus, obviously, $\varphi \subset c_A$ is true. We assume $\sup_{n\, k \in \mathbb{N}} |a_{nk}| = \infty$. Then we may choose sequences (n_j) and (k_j) in $\mathbb{N}$ with $k_j < k_{j+1}$ $(j \in \mathbb{N})$ such that

$$|a_{n,k_j}| \geq j^2 \qquad (j \in \mathbb{N})$$

Defining $y = (y_k)$ by

$$y_k := \begin{cases} \frac{1}{j^2} \operatorname{sgn} a_{n,k_j} & \text{if } k = k_j \\ 0 & \text{otherwise} \end{cases}$$

we obviously get

$$\Big| \sum_{k=k_j}^{k_j} a_{n,k} y_k \Big| = |a_{n,k_j}| \frac{1}{j^2} \geq 1 \qquad (j \in \mathbb{N}).$$

The last estimation gives us $y \notin S_A$ that contradicts $\ell \subset S_A$. □

In the next step we use this method to reprove both the well-known Schur theorem and the Hahn theorem. (The Schur theorem characterizes the matrices summing all bounded sequences, the Hahn theorem tells us that a conservative matrix which sums all $x \in \chi$ sums also all bounded sequences where χ denotes the set of all sequences with 0 and 1.) Moreover, we take an extended version of Schur's theorem (see [3]) into consideration.

THEOREM 4.4 (Extended theorem of Schur, theorem of Hahn). Let $A = (a_{nk})$ be a matrix. Then the following statements are equivalent:

(a) $m \subset c_A$.

(a*) $m \subset S_A$.

(b) $\exists\, \mu = (\mu_k), 0 < \mu_k \nearrow \infty : m_\mu \subset c_A$.

(b*) $\exists\, \mu = (\mu_k), 0 < \mu_k \nearrow \infty : m_\mu \subset S_A$.

(c) $\chi \subset c_A$, that is $m_0 \subset c_A$.

(c*) $\chi \subset S_A$, that is $m_0 \subset S_A$.

(d) $\varphi \subset c_A$ and $\sum_{k=1}^{\infty} |a_{nk}|$ converges uniformly in $n \in \mathbb{N}$.

(d*) $c_0 \subset c_A$ and $\limsup_n \sum_{k=1}^{\infty} |a_{nk} - a_k| = 0$ where a_k denotes the limit of the k–th column.

(e) $\varphi \subset c_A$ and $\exists\, \mu = (\mu_k), 0 < \mu_k \nearrow \infty : \sum_{k=1}^{\infty} \mu_k |a_{nk}|$ converges uniformly in $n \in \mathbb{N}$.

(e*) $c_0 \subset c_A$ and $\exists\, \mu = (\mu_k), 0 < \mu_k \nearrow \infty : \limsup_n \sum_{k=1}^{\infty} \mu_k |a_{nk} - a_k| = 0$.

Thereby, we can choose in (b), (b*) and (e) a common sequence μ.

REMARK 4.5. Originally, Schur proved '(a)$\Leftrightarrow$(d*)' and $\lim_A x = \sum_{k=1}^{\infty} a_k x_k$ $(x \in m)$ if (a) or (d*) in 4.4 is valid.

In case of conservative matrices the equivalence (a)$\Leftrightarrow$(c) is Hahn's theorem.

PROOF of 4.4. We are going to check the following chain of implications:

$$(b) \overset{(1)}{\Rightarrow} (b^*) \overset{(2)}{\Rightarrow} (a^*) \overset{(3)}{\Rightarrow} (a) \overset{(4)}{\Rightarrow} (c) \overset{(5)}{\Rightarrow} (c^*) \overset{(6)}{\Rightarrow} (d) \overset{(7)}{\Leftrightarrow} (d^*) \overset{(8)}{\Rightarrow} (e^*) \overset{(9)}{\Leftrightarrow} (e) \overset{(10)}{\Rightarrow} (b).$$

The implications (2), (3) and (4) and the equivalences (7) and (9) are obviously true.

The implications (1) and (5) are immediate corollaries of Theorem 3.5 since m_μ and m_0 have the p_wghp.

For a proof of (8) and (10) we refer to [3].

Now, we give a proof of (6). For that we assume that A is a matrix with real entries. [In the general case of complex entries we have to note that $\sum_k |a_{nk}|$ converges uniformly in $n \in \mathbb{N}$ if and only if this is true for the real part of a_{nk} and the imaginary part of a_{nk}.]

Let (c^*) be true. Then $\varphi \subset c_A$.

If $\sum_{k=1}^{\infty} |a_{nk}|$ does not converge uniformly in $n \in \mathbb{N}$ then we may choose an $\eta > 0$, a sequence (n_j) in $\mathbb{N}$ and index sequences (α_j) and (β_j) with $\alpha_j \leq \beta_j < \alpha_{j+1}$ such that

$$\sum_{k=\alpha_j}^{\beta_j} |a_{n_j k}| \geq \eta \qquad (j \in \mathbb{N}).$$

We define $y \in m_0$ by

$$y_k := \begin{cases} \operatorname{sgn} a_{n_j k} & \text{if } \alpha_j \leq k \leq \beta_j \\ 0 & \text{otherwise} \end{cases}$$

Since

$$\Big| \sum_{k=\alpha_j}^{\beta_j} a_{n_j k} y_k \Big| = \sum_{k=\alpha_j}^{\beta_j} |a_{n_j k}| \geq \eta \qquad (j \in \mathbb{N})$$

the series $\sum_{k=1}^{\infty} a_{nk} y_k$ does not converge uniformly in $n \in \mathbb{N}$. Therefore $y \notin S_A$ which contradicts $\chi \subset S_A$. □

5. EXAMPLES.

The aim of this section is the presentation of some examples distinguishing almost all of the gliding hump properties. For that purpose we collect known connections between gliding hump and related properties of sequence spaces in the following graphic.

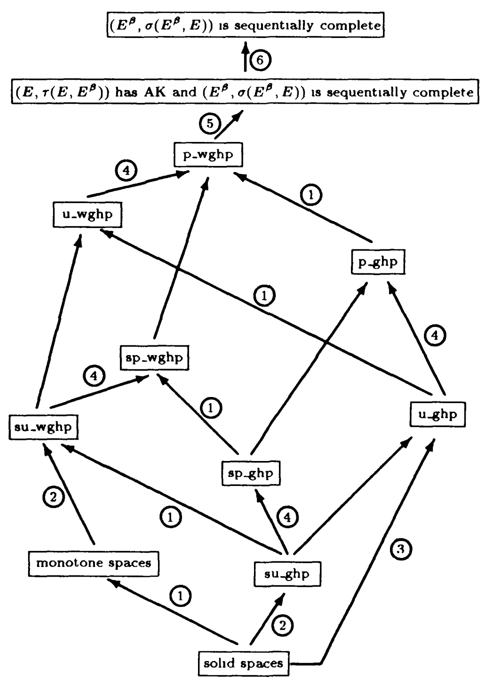

Figure 1:
Each arrow stands for 'implies' and the corresponding number in the circle gives the number of the example in 5.1 proving the strictness of the implication.

EXAMPLES 5.1 . (1) m_0 is a monotone space, thus it has all of the weak gliding hump properties ??_wghp. However, it does not have the p_ghp, thus no ??_ghp and it is not solid.

(2) The sequence space f_0 of all sequences almost convergent to 0 has all of the gliding hump properties. Furthermore, it is not a monotone space.
For a proof of the first statement we may prove that f_0 has the su_ghp by modifying Snyders proof of [20, Theorem7].

(3) $E := \ell_\mu \cap (c_0)_{c_1}$ with $\mu_k = k^2$ $(k \in \mathbb{N})$ has the u_ghp since $\mathcal{M}(E)$ has the gliding hump property (see [23, Theorem 3.3 and 3.4]). We don't know whether E has the su_ghp. Therefore, it may be a candidate to distinguish the properties su_ghp and u_ghp.

(4a) $E := \ell_2 \cap cs$ has the sp_ghp (thus all ?p_?ghp) since it is an FK–AK-space (Theorem 3.3). Furthermore, with [17, Corollary 4.4] we get that E is a sum space. Thus, by definition of a sum space $\mathcal{M}(E) := E^f$. Therefore, $\mathcal{M}(E) = E^f = \ell_2^f + cs^f = \ell_2 + bv \subset c$. From this and the fact that $e \in \mathcal{M}(E)$ we may derive that E cannot have the u_wghp (thus ?u_?ghp).

(4b) Considering the James space we get further sequence spaces having the same gliding humps properties as the example in (4a). For that let ω be the space of all real sequences and let

$$N(x) = \sup \left[\sum_{i=1}^{n} (x_{p_{2i-1}} - x_{p_{2i}})^2 + x_{p_{2n+1}}^2 \right]^{\frac{1}{2}}$$

where the supremum is taken over all positive integers n and all finite increasing sequences of integers $p_1, \ldots, p_{2n+1}$. Then

$$S_N = \left\{ x \in \omega \;\middle|\; N(x) < \infty \right\}$$

(together with its natural norm N) is a BK–space and the closure $J = S_N^\circ$ of φ in S_N is called James space (see [16]). We'll make use of the following facts:

(i) S_N is a BK-algebra with identity e.

(ii) $J = S_N \cap c_0$.

(iii) (e^k, E_k) is a shrinking basis for J^f so, in particular, J^f is AK thus AD.

(iv) $S_N = J \oplus \langle e \rangle = J^{ff}$.

Now by (i) and (iv) we get $\mathcal{M}(J^{ff}) = J^{ff}$ and by (iii) we get $\mathcal{M}(J^f) = \mathcal{M}(J^{ff})$ (see [7, Proposition 3.4]). Therefore by (ii) and (iv) we have

$$\mathcal{M}(J^f) = \mathcal{M}(J^{ff}) = J^{ff} = J \oplus \langle e \rangle \subset c. \tag{$*$}$$

As in (4a) we conclude that J^f has the sp_ghp (thus all ?p_?ghp) since it is an FK–AK–space. Furthermore, by $(*)$ and $e \in \mathcal{M}(J^f)$ we get that J^f cannot have the u_wghp (thus ?u_?ghp).
From $(*)$ we know that J^f is a sum space. Thus by [17, Corollary 4.4] $J^f \cap E$ will be a sum space too if E is any FK–space with unconditional basis (e^k). Then

$$\mathcal{M}(J^f \cap E) = (J^f \cap E)^f = J^{ff} + E^f = (J \oplus \langle e \rangle) + E^f$$

Now, let E be any FK–space with unconditional basis (e^k) such that $E^f \subset c$. Then, as above we may conclude, $J^f \cap E$ has all ?p_?ghp and no ?u_?ghp.

(5) Obviously, bs does not have the p_wghp thus none of the gliding hump properties in consideration. However it is known that $(bs, \tau(bs, bv_0))$ is an AK space and $(bv_0, \sigma(bv_0, bs))$ is sequentially complete.

(6) Let F be any separable FK space with $S_F \subsetneq W_F$ (for example, the domain of a conull matrix being not strongly conull) and let $E := W_F$. Then $(E^\beta, \sigma(E^\beta, E))$ is sequentially complete (see [1, Theorem 1 and 2]) but $(E, \tau(E, E^\beta))$ is not an AK-space since otherwise from Theorem 3.6 we would get $S_F \supset E = W_F$ thus $S_F = W_F$. □

Closing the paper we mention, that we don't know whether there is a difference between the s?_?ghp and the corresponding ?_?ghp (see Figure 1 and Example 5.1(3)).

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