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STABILITY OF NONLINEAR SYSTEMS UNDER CONSTANTLY ACTING PERTURBATIONS

XINZHI LIU

Department of Applied Mathematics
University of Waterloo
Waterloo, Ontario N2L 3G1, Canada

S. SIVASUNDARAM

Department of Mathematics and Physical Sciences
Embry-Riddle Aeronautical University
Daytona Beach, FL 32114, U.S.A.
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ABSTRACT. In this paper, we investigate total stability, attractivity and uniform stability in terms of two measures of nonlinear differential systems under constant perturbations. Some sufficient conditions are obtained using Lyapunov's direct method. An example is also worked out.

KEY WORDS AND PHRASES. Stability, perturbation, Lyapunov function, two measures.
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1. INTRODUCTION

When we model a physical system by means of a differential equation, it is not generally possible to take into account all the causes which determine the evolution. In other words, we have to admit that there are small perturbations permanently acting which cannot be accurately estimated consequently the validity of the description of the evolution, as given by a corresponding solution of the differential equation, requires that this solution be "stable" not only with respect to the small perturbations of the initial conditions, but also with respect to the perturbations, small in a suitable sense, of the right hand side of the equation. This kind of stability is called total stability, which we shall define in the next section.

There are several different concepts of stability studied in the literature, such as eventual stability, partial stability, conditional stability, etc. To unify these varieties of stability notions and to offer a general basis for investigation, it is convenient to introduce stability in terms of two different measures. Following Movchan [4], Salvadori [5] has successfully developed the theory of stability in terms of two measures. In the recent years much work has been done using two measures. See [2,3] and references therein.

In this paper we investigate the total stability, attractivity and uniform stability of perturbed systems in terms of two measures. In view of the generality of the present approach, our results improve and include some of the earlier findings and may be suitable for many applications.

2. PRELIMINARIES

Consider the differential system

$$x' = f(t,x), \quad x(t_0) = x_0 \tag{2.1}$$

and the perturbed differential system

$$x' = f(t,x) + R(t,x), \quad x(t_0) = x_0\,, \tag{2.2}$$

where $f, R \in C[R_+ \times R^n, R^n]$, $R(t,x)$ is a perturbation term relative to unperturbed system (2.1).

Let us begin by defining the following class of functions for future use.

$$K = \{a \in C[R_+, R_+] : a(u) \text{ is strictly increasing and } a(0) = 0\};$$

$$\Gamma = \left\{h \in C[R_+ \times R^N, R_+], \quad \inf h(t,x) = 0\right\}.$$

Definition 2.1 Let h_0, $h \in \Gamma$, then we say that h_0 is uniformly finer than h if there exists a $\delta > 0$ and a function $\Phi \in K$ such that

$$h_0(t,x) < \delta \quad \text{implies} \quad h(t,x) \leq \Phi(h_0(t,x)).$$

Definition 2.2 Let $V \epsilon C[R_+ \times R^N, R_+]$ and h_0, $h \in \Gamma$ then $V(t,x)$ is said to be

(i) h-positive definite if there exist a $\rho > 0$ and a function $b \in K$ such that $h(t,x) < \rho$ implies $b(t(t,x)) \leq V(t,x)$.

(ii) h_0 - decrescent if there exist a $\delta > 0$ and a function $a \in K$ such that

$$h_0(t,x) < \delta \quad \text{implies} \quad V(t,x) \leq a(h_0(t,x)).$$

Let $h_0, h \in \Gamma$. We shall now define the stability concepts for the system (2.1) in terms of two measures (h_0, h). Let $S(h,\rho) = \left[(t,x) \in R_+ \times R^n, \ h(t,x) < \rho\right]$.

Definition 2.3 The system (2.1) is said to be (h_0, h, T_1) - totally stable, if given $\epsilon > 0$ and $t_0 \epsilon R_+$, there exist two numbers δ_1, $\delta_2 > 0$ such that $h_0(t_0, x_0) < \delta_1$ and

$$\| R(t,x) \| < \delta_2 \quad \text{for} \quad (t,x) \in S(h,\epsilon) \tag{2.3}$$

imply $h(t, y(t,t_0,x_0)) < \epsilon$, $t \geq t_0$, where $y(t,t_0,x_0)$ is any solution of the perturbed system (2.2).

Definition 2.4 The system (2.1) is said to be (h_0, h, T_2) - totally stable, if for every $\epsilon > 0$, $t_0 \in R^+$ and $T > 0$ there exist two positive numbers $\delta_1 = \delta_1(\epsilon)$ and $\delta_2 = \delta_2(\epsilon)$ such that for every solution $y(t) = y(t,t_0,x_0)$ of (2.2) the inequality $h(t,y(t)) < \epsilon$, $t \geq t_0$ satisfies, provided that $h_0(t_0,x_0) < \delta_1$, $\| R(t,x) \| \leq \mu(t)$ for $h(t,x) \leq \epsilon$ and $\int_t^{t+T} \mu(s)ds < \delta_2$.

Definition 2.5 The system (2.1) is said to be (h_0, h) - attractive if given $t_0 \in R_+$, there exist a positive constant $\delta_0 = \delta(t_0)$ such that $h_0(t_0,x_0) < \delta_0$ implies $\lim_{t\to\infty} h(t,x)) = 0$.

We need the following known results [1] for our discussion.

Lemma 2.1 Let $g \in C[R_+ \times R, R]$ and $r(t) = r(t,t_0,x_0)$ be maximal solution of

$$u' = g(t,u), \ u(t_0) = u_0 \geq 0 \tag{2.4}$$

existing on J. Suppose that $m \in [R_+, R_+]$ and $Dm(t) \leq g(t, m(t))$, $t \in J$ where D is any fixed Dini derivative. Then $m(t_0) \leq u_0$ implies $m(t) \leq r(t)$, $t \in J$.

3. MAIN RESULTS

In this section we shall investigate the stability and attractivity properties of the differential system.

THEOREM 3.1 Assume that

(i) h_0, $h \in \Gamma$ and h_0 is uniformly finer than h.

(ii) $V \in C^1[R_+ \times R^n, R_+]$, $V(t,x)$ is h-positive definite, h_0-decrescent and

$$V'_{2.2}(t,x) \leq -C(h_0(t,x)), \quad (t,x) \in S(h,\rho), \quad C \epsilon K.$$

(iii) $\| V(t,x) - V(t,y) \| \leq M \| x - y \|$, $(t,x), (t,y) \in S(h,\rho)$ and $M > 0$.

Then the system (2.1) is (h_0, h, T_1) - totally stable.

PROOF: Let us write $V'_{22}(t,y)$, the time derivative of V along the solutions of the perturbed system (2.2). Then it follows from (ii) and (iii) that

$$V'_{22}(t,y) \le -C(h_0(t,y)) + M \parallel R(t,y) \parallel, (t,y) \in S(h,\rho) \tag{3.1}$$

Since $V(t,x)$ is h-positive definite and h_0-decrescent, there exist constants $\rho_0 \in (0,\rho)$, $\delta_0 > 0$ and functions $a, b \in K$ such that

$$V(t,x) \le a(h_0(t,x)) \quad \text{if} \quad h_0(t,x) < \delta_0 \tag{3.2}$$

and

$$b(h(t,x)) \le V(t,x) \quad \text{whenever} \quad h(t,x) < \rho_0 . \tag{3.3}$$

Let $\epsilon \in (0,\rho_0)$ be given. Choose $\delta_1 \in (0,\delta_0)$ such that

$$a(\delta_1) < b(\epsilon) \quad \text{and} \quad h(t,x) < \epsilon \quad \text{if} \quad h_0(t,x) < \delta \tag{3.4}$$

because of the assumptions on a, b and condition (i).

For $K \in (0,1)$, choose $\delta_2 = K\dfrac{C(\delta_1)}{M}$. Let $t_0 \in R_+$ and $y(t) = y(t,t_0,x_0)$ be a solution of (2.2) we claim that $h(t_0,x_0) < \delta$, and $\parallel R(t,y) \parallel < \delta_2$ for $(t,y) \in S(h,\epsilon)$ implies

$$h(t,y(t)) < \epsilon, \quad t \ge t_0 . \tag{3.5}$$

If this is not true, there would exist a solution $y(t) = y(t,t_0,x_0)$ of (2.2) with $h_0(t_0,x_0) < \delta_1$ and $t_2 > t_1 > t_0$ such that

$$h_0(t_1,y(t_1)) = \delta_1, \quad h(t_2,y(t_2)) = \epsilon, (t,y(t)) \in S(h,\epsilon) \cap S^c(h_0,\delta_1) \tag{3.6}$$

and

$$\parallel R(t,y(t)) \parallel < \delta_2, \quad t \in [t_1,t_2) .$$

Then it follows from (3.1) and (3.6) that

$$V'(t,y(t)) \le -C(\delta_1) + MK\frac{C(\delta_1)}{M} < 0, \quad t_1 \le t \le t_2 .$$

which implies by (3.2)-(3.4) that

$$b(\epsilon) \le V(t_2,y(t_2)) \le V(t_1,y(t_1)) \le a(\delta_1) < b(\epsilon) .$$

This contradiction shows that (3.5) is true and thus the system (2.1) is (h_0,h,T_1) - totally stable which completes the proof of the theorem.

THEOREM 3.2 In addition to the assumption of theorem 3.1, suppose further that there exist a constant $\sigma > 0$ such that $h(t,x) < \sigma$ implies

$$\lim_{t\to\infty} R(t,y) = 0 \tag{3.7}$$

uniformly in y.

Then the system (2.2) is (h_0,h)-attractive.

PROOF: Because of (h_0,h) - total stability of system (2.1), setting $\epsilon = \sigma_0 = \min\{\rho_0,\sigma\}$, there exists constants $\delta_{10} > 0$ and $\delta_{20} > 0$ such that $h_0(t_0,x_0) < \delta_{10}$ and $\parallel R(t,x) \parallel < \delta_{20}$ for $(t,x) \in S(h,\sigma_0)$ implies

$$h(t,y(t)) < \sigma_0, \quad t \ge t_0 , \tag{3.8}$$

where $y(t) = y(t,t_0,x_0)$ is any solution of (2.2).

Let $\eta \in (0,\sigma_0)$ and $\delta_1 = \delta_1(\eta)$, $\delta_2 = \delta_2(\eta)$ be chosen as in the definition 2.3. Let $\sigma_2^* = \min\left\{\delta_2, \dfrac{C(\delta_1)}{M}\right\}$, it follows from (3.7) that there exist $T_1 = T(t_0,x_0) > 0$ such that

$$\| R(t, y(t)) \| < \delta_2^*, \quad t \geq T + t_0 . \tag{3.9}$$

To show (h_0, h)-attractivity of (2.2), it is enough to prove that there exists a $T = T(t_0, x_0) > 0$ such that for some $t^* \in [t_0, T + t_0]$

$$h_0(t^*, x(t^*)) < \delta_1 \quad \text{and} \quad \| R(t, y(t)) \| < \delta_2^*, \; t \geq t^* .$$

Choose

$$T = \frac{4a(h_0(t_0 + T_1, y(t_0 + T_1)))}{C(\delta_1)} + T_1 .$$

Then if for $t_0 + T_1 \leq t \leq t_0 + T$, $(t, y(t)) \in S(h, \sigma_0) \cap S^c(h_0, \delta_1)$, we get by (3.1) and (3.2)

$$V'(t, y(t)) \leq -\frac{C(\delta_1)}{2}, \quad t_0 + T_1 \leq t \leq t_0 + T .$$

which implies

$$V(t_0 + T, y(t_0 + T)) \leq a(h_0(t_0 + T_1, y(t_0 + T_1))) - \frac{C(\delta_1)}{2}(T - T_1) < 0 .$$

This contradiction shows the existence of t^* and it follows from (h_0, h, T_1) - total stability of (2.1) that the system (2.2) is (h_0, h) - attractive, which completes the proof of the theorem.

The next result is on (h_0, h, T_2) - stability.

THEOREM 3.3 Assume that

(i) $h_0, h \in \Gamma$ and h_0 is uniformly finer than h.

(ii) $V \in C^1[R_+ \times R^n, R_+]$, $V(t, x)$ is h-positive definite $V(t, x)$ is h_0-decrescent and

$$V'_{2.1}(t, x) \leq -C(V(t, x)), \quad (t, x) \epsilon S(h, \rho), \quad C \in K .$$

(iii) $\| V(t, x) - V(x, y) \| \leq M \; \| x - y \|$, $(t, x), (t, y) \in S(h, \rho)$ and $M > 0$.

Then the system (2.1) is (h_0, h, T_2) - totally stable.

PROOF: Using the relations (3.1) and (3.2) we choose $\delta_1 = \delta_1(\epsilon)$ such that

$$a(\delta_1) < b(\epsilon) . \tag{3.10}$$

Let $h_0(t_0, x_0) < \delta_1$, $m(t) = V(t, y(t))$, where $y(t) = y(t, t_0, x_0)$ is a solution of (2.2). Hence $m(t_0) < a(\delta_1) < b(\epsilon)$. We claim $m(t) < b(\epsilon)$, $t \geq t_0$. If this is not true, then there exist a $t_1 > t_0$ such that $m(t_1) = b(\epsilon)$ and $m(t) \leq b(\epsilon)$ for $t_0 \leq t_1$ which implies

$$h(t, y(t)) \leq \epsilon < \rho , \quad t_0 \leq y \leq t_1 \tag{3.11}$$

Let $t_1 - t_0 = T$ and choose

$$\delta_2 = \delta_2(\epsilon) < b(\epsilon) - J^{-1}\{J(a(\delta_1))\}/M \tag{3.12}$$

where

$$J(u) - J(u_0) = \int_{u_0}^{u} \frac{ds}{C(s)}, \quad J(u) = \int_0^u \frac{dS}{C(s)} \quad \text{if} \quad \int_0^u \frac{ds}{C(s)} < \infty$$

Otherwise $J(u) = \int_\delta^u \frac{ds}{C(s)}$ for some small constant $\delta > 0$, and J^{-1} is the inverse function of J.

From (ii) and (iii) we have

$$D^+V(t, y(t)) \leq -C\big(V(t, y(t))\big) + M \; \| R(t, y(t)) \|$$

for $t \in [t_0, t_1]$.

Now define $\lambda(t) = V(t, y(t)) - \gamma(t)$, where

$$\gamma(t) = M \int_{t_0}^{t} \| R(s, y(s)) \| ds .$$

We obtain

$$D^+\lambda(t) \leq -C(\lambda(t))$$

using the monotonic character of $C(u)$ and the fact $\lambda(t) \geq V(t, y(t))$, which implies, by Lemma 2.1, that

$$\lambda(t) \leq J^{-1}\Big[J(V(t_0, x_0)) - (t - t_0)\Big], \quad t \in [t_0, t_1\} \tag{3.13}$$

Noting that the maximal solution of $u' = -C(u)$, $u(t_0) = V(t_0, x_0)$, is just the right hand side of (3.13), thus it follows

$$V(t, y(t)) \leq J^{-1}\Big[J(V(t_0, x_0)) - (t - t_0)\Big] + \gamma(t), \quad t \in [t_0, t_1].$$

Now using the facts that $h(t, y(t)) \leq \epsilon$ for $t_0 \leq t \leq t_0 + T$, $V(t_0, x_0) < a(\delta_1)$, $\| R(t, y(t)) \| \leq \mu(t)$, $\int_t^{t+T} \mu(s)ds < \delta_2$ and relations (3.10), (3.12), we derive the inequality

$$b(\epsilon) \leq V(t_0 + T, y(t_0 + T)) \leq J^{-1}\Big[J(a(\delta_1)) - T\Big] + M\delta_2 < b(\epsilon),$$

which is a contradiction. Thus $m(t) < b(\epsilon)$, $t \geq t_0$, which implies (h_0, h, T_2)-total stability of (2.1). This completes the proof of the theorem.

In the previous theorems, in order to prove total stability properties of (2.2) we assumed the uniform asymptotic stability properties of (2.1) (the unperturbed system). In the following theorem we prove (h_0, h) - stability of (2.2) under weaker assumptions on (2.1) by avoiding using norm on the perturbed term.

THEOREM 3.4 Assume that

(i) $h_0, h \in \Gamma$ and h_0 is uniformly finer than h.

(ii) $V \in C^1[R_+ \times R^n, R_+]$, $V(t,x)$ is h-positive definite, h_0-decrescent and

$$V'_{2.1}(t,x) \leq 0, \quad (t,x) \in S(h, \rho).$$

(iii) $\dfrac{\partial V(t,x)}{\partial x} \cdot R(t,x) \leq \ell(t)V(t,x)$, $(t,x) \in S(h,\rho)$,
where $\ell(t) \in \mathcal{L}^1$ and $\exp\left[\int_{t_1}^{t_2} \ell(s)ds\right] \leq M$, $M > 0$.
Then the system (2.2) is (h_0, h)-uniformly stable.

PROOF: Let us write $V'_{2.2}(t,y)$, the time derivative of V along the solutions of the perturbed system (2.2). Then it follows from (ii) and (iv) that

$$V'_{2.2}(t,y) \leq V'_{2.1}(t,y) + \frac{\partial V(t,x)}{\partial x} \cdot R(t,x) \leq \ell(t)V(t,y), \; (t,y) \in S(h,\rho) \tag{3.14}$$

Since $V(t,x)$ is h-positive definite and h_0-decrescent, there exist constants $\rho_0 \in (0,\rho)$, $\delta_0 > 0$ and functions a, $b \epsilon K$ such that

$$V(t,x) \leq a\,(h_0(t,x)) \quad \text{if} \quad h_0(t,x) < \delta_0 \tag{3.15}$$

and

$$b(h(t,x)) \leq V(t,x) \quad \text{whenever} \quad h(t,x) < \rho_0 \tag{3.16}$$

Let $\epsilon \in (0, \rho_0)$ be given. Because of the assumptions of a, b and condition (i) we can choose $\delta_1 \in (0, \delta_0)$ such that

$$M\,a(\delta_1) < b(\epsilon) \quad \text{and} \quad h(t,x) < \epsilon \quad \text{if} \quad h_0(t,x) < \delta_1 . \tag{3.17}$$

Let $t_0 \in R_+$ and $y(t) = y(t, t_0, x_0)$ be a solution of (2.2). We claim that $h(t_0, x_0) < \delta_1$ implies

$$h(t, y(t)) < \epsilon, \quad t \geq t_0 \quad \text{for} \quad (t, y) \in S(h, \epsilon) \tag{3.18}$$

If this is not true, there would exist a solution $y(t) = y(t, t_0, x_0)$ of (2.2) with $h_0(t_0, x_0) < \delta_1$ and $t_2 > t_1 > t_0$ such that $h_0(_1, y(t_1)) = \delta_1$, $h(t_2, y(t_2)) = \epsilon$ and

$$(t, y(t)) \epsilon S(h, \in) \cap S^c(h_0, \delta_1), \quad t \in [t_1, t_2), \tag{3.19}$$

then it follows from (3.14)

$$V'(t, y(t)) \leq \ell(t) V(t, y(t))$$

which implies by (3.15)-(3.18)

$$b(\epsilon) \leq V(t_2, y(t_2) \leq V(t_1, y(t_1)) \exp\left[\int_{t_1}^{t_2} \ell(s) ds\right] \leq M a(\delta_1) < b(\epsilon).$$

This contradiction shows that (3.18) is true, which completes the proof of the theorem.

To conclude our paper, we consider the following example.

EXAMPLE: Consider the differential system

$$\begin{aligned} x_1' &= x_2 + (1 - x_1^2 - x_2^2) x_1 e^{-t} \\ x_2' &= x_1 + (1 - x_1^2 - x_2^2) x_2 \sin x_1^2 \end{aligned} \tag{3.20}$$

and the perturbed system

$$\begin{aligned} x_1' &= -x_2 + (1 - x_1^2 - x_2^2) x_1 e^{-t} + R_1(t, x_1, x_2) \\ x_2 &= x_1 + 1(1 - x_1^2 - x_2^2) x_2 \sin x_1^2 + R_2(t, x_1, x_2) \end{aligned} \tag{3.21}$$

where

$$R_1(t, x_1, x_2) = (x_1^2 + x_2^2 - 1) t^2 \frac{e^t}{x_1},$$

and

$$R_2(t, x_1, x_2) = (x_1^2 + x_2^2 - 1) \frac{e^{-t/2} \sin t}{x_2}.$$

Let $V(t, x) = (x_1^2 + x_2^2 - 1)^2$, $h_0 = h = | x_1^2 + x_2^2 - 1 |$. Then we see that

$$h^2(t, x) \leq V(t, x) \leq h_0^2(t, x)$$

and

$$V_{4.1}'(t, x) = -(x_1^2 + x_2^2 - 1)^2 (x_1^2 e^{-t} + x_2^2 \sin x_1^2) \leq 0, \quad (t, x) \in R_+ \times R^2,$$

$$\frac{\partial V}{\partial x} \cdot R(t, x) = \frac{\partial V}{\partial x_1} R_1(t, x_1, x_2) + \frac{\partial V}{\partial x_2} R_2(t, x_1, x_2) \leq \ell(t) V(t, x),$$

where $\ell(t) = 4[t^2 e^{-t} + \sin t e^{-t/2}]$. Hence by Theorem 3.4, the perturbed system (4.2) is (h_0, h)-uniformly stable.

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