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CLASSICAL QUOTIENT RINGS OF GENERALIZED MATRIX RINGS

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ABSTRACT. An associative ring R with identity 1 is a generalized matrix ring with idempotent set E if E is a finite set of orthogonal idempotents of R whose sum is 1. We show that, in the presence of certain annihilator conditions, such a ring is semiprime right Goldie if and only if eRe is semiprime right Goldie for all $e \in E$, and we calculate the classical right quotient ring of R.

KEY WORDS AND PHRASES. Generalized matrix ring, quotient ring, Goldie conditions.
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1. INTRODUCTION.

Each ring considered in this paper is associative and has an identity. Such a ring R is a *generalized matrix ring with idempotent set* E if E is a finite set of orthogonal idempotents of R whose sum is 1.

In this paper, we show that, in the presence of certain non-degeneracy conditions, a generalized matrix ring R with idempotent set E is semiprime right Goldie if and only if eRe is semiprime right Goldie for all $e \in E$, and we calculate the classical right quotient ring of R. Kerr's example [4] of a right Goldie ring whose matrix ring is not right Goldie shows that our semiprimeness condition cannot be omitted.

Examples of generalized matrix rings include incidence algebras of directed graphs with a finite number of vertices (see [5] and [9]), structural matrix rings (see Van Wyk [13] and subsequent papers), endomorphism rings of finite direct sums of modules and Morita context rings. Sands [10] observed that if $[S, V, W, T]$ is a Morita context, then

$$\begin{bmatrix} S & V \\ W & T \end{bmatrix}$$

is a ring. These Morita context rings are precisely generalized matrix rings with idempotent sets E such that $|E| = 2$, and they have been widely studied. In particular, we note Amitsur's paper [1], the survey paper [6], McConnell and Robson's treatment in ([7], 1.1 and 3.6) and

Müller's computation of the maximal quotient ring in [8].

A generalized matrix ring R with idempotent set E is called a piecewise domain if for all $e, f, g \in E$, $x \in eRf$ and $y \in fRg$, we have $xy = 0$ implies $x = 0$ or $y = 0$. These rings have been studied in some detail – see, for instance, [2] and [3].

We denote the prime radical of a ring R by $p(R)$ and if e and f are idempotents of R, $e \neq f$, $p(eRf)$ denotes the set $\{x \in eRf: xfRe \subseteq p(eRe)\}$.

PROPOSITION 1.1 (Sands [10]). If R is a generalized matrix ring with idempotent set E, then

$$p(R) = \sum_{e,f \in E} p(eRf).$$

PROOF. If $|E| = 1$ there is nothing to prove and if $|E| = 2$ this is Theorem 1 in Sands [10]. Assume now that $|E| = n > 2$ and that the theorem is true for generalized matrix rings with idempotent sets of cardinality less than n. Let $e \in E$ and set $E' = \{e, 1-e\}$. Then R is a generalized matrix ring with idempotent set E' and so Sands' result implies that

$$p(R) = p(eRe) + p(eR(1-e)) + p((1-e)Re) + p((1-e)R(1-e)).$$

Since $(1-e)R(1-e)$ is a generalized matrix ring with idempotent set $E_1 = E\backslash\{e\}$, our induction hypothesis implies that

$$p((1-e)R(1-e)) = \sum_{f,g \in E_1} p(fRg).$$

Also, it is clear that $p(eR(1-e)) = \sum_{f \in E_1} p(eRf)$ and $p((1-e)Re) = \sum_{f \in E_1} p(fRe)$, so the result follows. □

Let R be a generalized matrix ring with idempotent set E. We say that the pair (R, E) satisfies the *left* (respectively, *right*) *annihilator condition* if for all $e, f \in E$, $0 \neq x \in eRf$ implies that $xfRe \neq 0$ (respectively, $fRex \neq 0$). This concept is defined in [12] where right and left are interchanged.

COROLLARY 1.2. (Wauters and Jespers [12]). The following conditions on a generalized matrix ring with idempotent set E are equivalent.

(a) R is semiprime.

(b) (R, E) satisfies the left annihilator condition and eRe is semiprime for all $e \in E$.

(c) (R, E) satisfies the right annihilator condition and eRe is semiprime for all $e \in E$. □

2. THE GOLDIE CONDITIONS.

The right singular ideal of a ring S will be denoted by $Z(S)$, and the right singular submodule of a right S-module M will be denoted by $Z(M)$. So, if R is a generalized matrix ring with idempotent set E and $e, f \in E$ with $e \neq f$, then $Z(eRe)$ is the right singular ideal of the ring eRe and $Z(eRf)$ is the right singular submodule of the right fRf-module eRf.

PROPOSITION 2.1. Let R be a generalized matrix ring with idempotent set E and suppose that (R, E) satisfies the left annihilator condition. Then

$$Z(R) = \sum_{e,f \in E} Z(eRf).$$

PROOF. Let $e, f \in E$ and suppose that $x \in Z(R)$. Then $exf \in Z(R)$, so there is an essential right ideal I of R such that $exfI = 0$. To show that $exf \in Z(eRf)$ it suffices to show that fIf is an essential right ideal of fRf. Let A be a nonzero right ideal of fRf. Because I is essential,

$I \cap AR \neq 0$. Let $0 \neq u \in I \cap AR$. Since $I \cap AR$ is a right ideal there is an idempotent $g \in E$ such that $0 \neq ug \in I \cap AR$, and $ug = fug$ because $ug \in AR \subseteq fRfR$. Since (R, E) satisfies the left annihilator condition, $0 \neq (fug)gRf \subseteq (I \cap AR) \cap fRf \subseteq fRf \cap A$. It follows that

$$Z(R) \subseteq \sum_{e,f \in E} Z(eRf).$$

Conversely, suppose that $e, f \in E$ and $y = eyf \in Z(eRf)$. Then $yH = 0$ for some essential right ideal H of fRf. Let $J = \{r \in R : fr \in HR\}$. Clearly, J is a right ideal of R and $yJ = eyfJ = (eyf)fJ \subseteq (eyf)HR = 0$, so to show that $y \in Z(R)$ it is enough to show that J is essential in R. Let B be a nonzero right ideal of R. If $fB = 0$, then $B \subseteq J$ and so $B \cap J \neq 0$. Now assume $fB \neq 0$. Then $fBg \neq 0$ for some $g \in E$, and so the left annihilator condition implies that $fBf \neq 0$. So we see that fBf is a nonzero right ideal of fRf. Thus $fBf \cap H \neq 0$ and so $B \cap J \neq 0$ because $H \subseteq HR$. □

COROLLARY 2.2. If R is a generalized matrix ring with idempotent set E such that (R, E) satisfies the left annihilator condition, then R is nonsingular if and only if eRe is nonsingular for all $e \in E$.

PROOF. In view of the proposition, we need only show that $Z(R) \neq 0$ implies that $Z(eRe) \neq 0$ for some $e \in E$. Suppose that $0 \neq x \in Z(R)$. Then $0 \neq exf \in Z(R)$ for some $e, f \in E$. The right annihilator condition implies that $(exf)fRe \neq 0$ and so $eRe \cap Z(R) \neq 0$. It now follows from the proposition that $Z(eRe) \neq 0$. □

The right uniform dimension of a ring R (respectively, right R-module M) will be denoted by $d(R)$ (respectively, $d(M)$).

PROPOSITION 2.3. Let R be a generalized matrix ring with idempotent set E such that (R, E) satisfies the left annihilator condition. If $d(R) < \infty$ then $d(eRe) < \infty$ for all $e \in E$. Moreover, if R is semiprime and $d(eRe) < \infty$ for all $e \in E$, then $d(eRf) < \infty$ for all $e, f \in E$ and hence $d(R) < \infty$.

PROOF. Assume that $d(R) < \infty, e \in E$ and $\sum A_i$ is a direct sum of nonzero right ideals of eRe. To prove that $d(eRe) < \infty$ it is enough to show that $\sum A_i R$ is direct, and to accomplish this we need only show that $\sum A_i Rf$ is direct for each $f \in E$. Suppose that $f \in E$ and $b_i \in A_i Rf$ are such that $\sum b_i = 0$. Since $b_i fRe \subseteq A_i$ and $\sum A_i$ is direct, $b_i fRe = 0$ for all i. Thus the left annihilator condition implies that $b_i = 0$ for all i and hence $\sum A_i Rf$ is direct.

Now assume that $d(eRe) < \infty$ for all $e \in E$ and suppose that $\sum N_i$ is a direct sum of nonzero fRf-submodules of eRf. Since $0 \neq N_i \subseteq eRf$ the left annihilator condition implies that $N_i fRe \neq 0$, and each $N_i fRe$ is a right ideal of eRe. Let $K = N_j fRe \cap \sum\{N_i fRe : i \neq j\}$. Then $KeRf \subseteq N_j \cap \sum\{N_i : i \neq j\}$ and so $KeRf = 0$. Since $K^2 \subseteq KeRf, K^2 = 0$ and so $K = 0$ because eRe is semiprime by Corollary 2. Thus $\sum N_i fRe$ is direct and so $d(eRf) \leq d(eRe)$. It follows that R has finite right uniform dimension as a right $\left(\sum_{e \in E} eRe\right)$-module and so certainly $d(R) < \infty$. □

From Corollary 2, Corollary 4 and Proposition 5 we obtain the following theorem.

THEOREM 2.4. Let R be a generalized matrix ring with idempotent set E. If R is semiprime right Goldie, then so too are the rings $eRe, e \in E$. Conversely, if (R, E) satisfies the left annihilator condition and eRe is semiprime right Goldie for all $e \in E$, then R is semiprime right Goldie.

3. THE QUOTIENT RING.

Let S and T be rings and let M be an $S-T-$bimodule. We say that M satisfies the *right* *bimodule Ore condition* if for each $m \in M$ and each regular element $c \in S$ there is an $m_1 \in M$ and a regular $c_1 \in T$ such that $mc_1 = cm_1$.

PROPOSITION 3.1. If R is a semiprime right Goldie generalized matrix ring with idempotent set E, then eRf satisfies the right bimodule Ore condition for all $e, f \in E$.

PROOF. Let $m \in eRf$ and suppose that c is regular in eRe. Define $\theta: eRf \to ceRf$ by $\theta(x) = cx$ for all $x \in eRf$. Clearly θ is an fRf-module homomorphism and we now check that θ is a monomorphism. Suppose that $x \in eRf$ and $cx = 0$. Then $cxfRe = 0$ which implies that $xfRe = 0$ because c is regular. But then the left annihilator condition implies that $x = 0$ as required.

From Theorem 6 and Proposition 5 we know that eRf has finite right uniform dimension as a right fRf-module. Since eRf and $ceRf$ are isomorphic fRf-modules, $d(ceRf) = d(eRf)$ and so $ceRf$ is an essential fRf submodule of eRf. Hence $\{y \in fRf: my \in ceRf\}$ is an essential right ideal of fRf which, since fRf is semiprime right Goldie, must contain the required regular element c_1. □

If S is semiprime right Goldie, $Q(S)$ denotes the classical right quotient ring of S and if M is a right S-module, $Q(M) = M \otimes_S Q(S)$. Using the right common denominator property of Ore sets we see that every element of $Q(M)$ is of the form $m \otimes c^{-1}$ where $m \in M$ and c is regular in S. In what follows we shall write mc^{-1} instead of $m \otimes c^{-1}$.

THEOREM 3.2. If R is a semiprime right Goldie generalized matrix ring with idempotent set E, then

$$Q(R) = \sum_{e,f \in E} Q(eRf).$$

PROOF. For each $e \in E, eRe$ embeds in $Q(eRe)$ and we now check that for $e, f \in E, e \neq f, eRf$ embeds in $Q(eRf)$. Suppose that c is regular in fRf, $x \in eRf$ and $xc = 0$. Then $fRexc = 0$ and so $fRex = 0$ because c is regular in fRf. Thus $exfRexf = 0$ and hence $0 = exf = x$ since R is semiprime. This shows that eRf is a torsion free fRf-module and so eRf embeds in $Q(eRf)$.

Let $e, f, g \in E$, $x \in eRf$, $y \in fRg$ and suppose that c is regular in fRf and d is regular in gRg. Define $(xc^{-1})(yd^{-1}) = xy_1c_1^{-1}d^{-1}$ where y_1 and c_1 are obtained from the right bimodule Ore condition: $yc_1 = cy_1$. It is straightforward to check that this multiplication is well-defined and that as a result $Q = \sum_{e,f \in E} Q(eRf)$ becomes a generalized matrix ring with idempotent set E.

We now show that Q is semiprime. It follows from Theorem 6 that eRe is semiprime right Goldie for all $e \in E$ and hence $Q(eRe)$ is semiprime for all $e \in E$. In view of Corollary 1.2, it suffices to show that (Q, E) satisfies the right annihilator condition. Let $yd^{-1} \in Q(fRe)$ be such that $Q(eRf)yd^{-1} = 0$. Then $(eRf)(yd^{-1}) = 0$ and so $(eRf)y = 0$. From Corollary 1.2 we see that (R, E) satisfies the right annihilator condition and so $y = 0$. Thus (Q, E) satisfies the right annihilator condition and hence Q is semiprime.

Let $e, f \in E, e \neq f$. From Proposition 2.3 we see that eRf has finite uniform dimension as a right fRf-module and so $Q(eRf)$ has finite uniform dimension as a right $Q(fRf)$-module. Since $Q(fRf)$ is semisimple Artinian it follows that $Q(eRf)$ is an Artinian $Q(fRf)$-module, and hence Q is right Artinian by an argument similar to ([7], 1.1.7). Since we have already seen that Q is semiprime, Q is a semisimple Artinian ring.

To complete the proof, we need only show that R is a right order in Q. Let $x \in Q$, $x = \sum_{e,f \in E} x(e,f)$ where $x(e,f) \in Q(eRf)$ for all $e, f \in E$. Using the right common denominator *bimodule Ore condition* if for each $m \in M$ and each regular element $c \in S$ there is an $m_1 \in M$ and a regular $c_1 \in T$ such that $mc_1 = cm_1$.

PROPOSITION 3.1. If R is a semiprime right Goldie generalized matrix ring with idempotent set E, then eRf satisfies the right bimodule Ore condition for all $e, f \in E$.

PROOF. Let $m \in eRf$ and suppose that c is regular in eRe. Define $\theta: eRf \to ceRf$ by $\theta(x) = cx$ for all $x \in eRf$. Clearly θ is an fRf-module homomorphism and we now check that θ is a monomorphism. Suppose that $x \in eRf$ and $cx = 0$. Then $cxfRe = 0$ which implies that $xfRe = 0$ because c is regular. But then the left annihilator condition implies that $x = 0$ as required.

From Theorem 6 and Proposition 5 we know that eRf has finite right uniform dimension as a right fRf-module. Since eRf and $ceRf$ are isomorphic fRf-modules, $d(ceRf) = d(eRf)$ and so $ceRf$ is an essential fRf submodule of eRf. Hence $\{y \in fRf : my \in ceRf\}$ is an essential right ideal of fRf which, since fRf is semiprime right Goldie, must contain the required regular element c_1. □

If S is semiprime right Goldie, $Q(S)$ denotes the classical right quotient ring of S and if M is a right S-module, $Q(M) = M \otimes_S Q(S)$. Using the right common denominator property of Ore sets we see that every element of $Q(M)$ is of the form $m \otimes c^{-1}$ where $m \in M$ and c is regular in S. In what follows we shall write mc^{-1} instead of $m \otimes c^{-1}$.

THEOREM 3.2. If R is a semiprime right Goldie generalized matrix ring with idempotent set E, then

$$Q(R) = \sum_{e, f \in E} Q(eRf).$$

PROOF. For each $e \in E, eRe$ embeds in $Q(eRe)$ and we now check that for $e, f \in E, e \neq f, eRf$ embeds in $Q(eRf)$. Suppose that c is regular in fRf, $x \in eRf$ and $xc = 0$. Then $fRexc = 0$ and so $fRex = 0$ because c is regular in fRf. Thus $exfRexf = 0$ and hence $0 = exf = x$ since R is semiprime. This shows that eRf is a torsion free fRf-module and so eRf embeds in $Q(eRf)$.

Let $e, f, g \in E$, $x \in eRf$, $y \in fRg$ and suppose that c is regular in fRf and d is regular in gRg. Define $(xc^{-1})(yd^{-1}) = xy_1c_1^{-1}d^{-1}$ where y_1 and c_1 are obtained from the right bimodule Ore condition: $yc_1 = cy_1$. It is straightforward to check that this multiplication is well-defined and that as a result $Q = \sum_{e, f \in E} Q(eRf)$ becomes a generalized matrix ring with idempotent set E.

We now show that Q is semiprime. It follows from Theorem 6 that eRe is semiprime right Goldie for all $e \in E$ and hence $Q(eRe)$ is semiprime for all $e \in E$. In view of Corollary 1.2, it suffices to show that (Q, E) satisfies the right annihilator condition. Let $yd^{-1} \in Q(fRe)$ be such that $Q(eRf)yd^{-1} = 0$. Then $(eRf)(yd^{-1}) = 0$ and so $(eRf)y = 0$. From Corollary 1.2 we see that (R, E) satisfies the right annihilator condition and so $y = 0$. Thus (Q, E) satisfies the right annihilator condition and hence Q is semiprime.

Let $e, f \in E, e \neq f$. From Proposition 2.3 we see that eRf has finite uniform dimension as a right fRf-module and so $Q(eRf)$ has finite uniform dimension as a right $Q(fRf)$-module. Since $Q(fRf)$ is semisimple Artinian it follows that $Q(eRf)$ is an Artinian $Q(fRf)$-module, and hence Q is right Artinian by an argument similar to ([7], 1.1.7). Since we have already seen that Q is semiprime, Q is a semisimple Artinian ring.

To complete the proof, we need only show that R is a right order in Q. Let $x \in Q$, $x = \sum_{e, f \in E} x(e, f)$ where $x(e, f) \in Q(eRf)$ for all $e, f \in E$. Using the right common denominator property we can find, for each $f \in E$, an $a_f \in fRf$ and elements $y(e,f) \in eRf$ such that for all $e \in E, x(e,f) = y(e,f)a_f^{-1}$. Let $y = \sum_{e,f \in E} y(e,f)$ and $z = \sum_{f \in E} a_f$. Then $x = yz^{-1}$ and so R is a right order in Q. □

Let R be a semiprime right Goldie ring with idempotent e. Clearly, R is a generalized matrix ring with idempotent set $E = \{e, 1-e\}$ and so it follows from Theorem 2.4 that eRe is semiprime right Goldie. This result seems to have been well-known for some time. Also, it follows from Theorem 3.2 that the classical right quotient ring of eRe is eQe where Q is the classical right quotient ring of R. This result is due to Small [11].

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