Internat. J. Math. & Math. Sci.
VOL. 18 NO. 2 (1995) 383-390



# SEQUENTIAL POINT AND INTERVAL ESTIMATION OF SCALE PARAMETER OF EXPONENTIAL DISTRIBUTION

Z. GOVINDARAJULU

University of Kentucky
Lexington, Kentucky, U.S.A.

(Received January 13, 1993)

**Abstract**. Sequential fixed-width confidence intervals are obtained for the scale parameter $\sigma$ when the location parameter $\theta$ of the negative exponential distribution is unknown. Exact expressions for the stopping time and the confidence coefficient associated with the sequential fixed-width interval are derived. Also derived is the exact expression for the stopping time of sequential point estimation with quadratic loss and linear cost. These are numerically evaluated for certain nominal confidence coefficients, widths of the interval and cost functions, and are compared with the second order asymptotic expressions.

Key words and phrases: Stopping time, exponential distribution, sequential estimation of scale parameter.
1991 AMS Subject classification code 62L12

# 1 Introduction and Preliminaries.

Starr and Woodroofe [1] have considered the risk efficient estimation of the scale parameter $\sigma$ when the location, $\theta$, is zero and studied some of the first order properties of the sequential procedure. Govindarajulu and Sarkar [2] have considered the risk-efficient estimation of $\sigma$ when $\theta$ is unknown and studied the second order properties of the stopping time and the regret. Govindarajulu [3] has studied the second order asymptotic properties of the fixed-width interval estimation procedure for $\sigma$ when $\theta$ is unknown. Mukhopadhyay [4] has considered risk efficient estimation of the mean of a negative exponential distribution. Here we derive exact expressions for the stopping time and confidence coefficient of the fixed-width interval estimation procedure and for the stopping time associted with point estimation with quadratic loss and linear cost, and compare them with the second order asymptotic expressions.

Let $X_1, X_2, \ldots,$ be an i.i.d. sequence of random variables having the density:

$$f(x:\theta,\sigma) = \sigma^{-1}\exp\{-(x-\theta)/\sigma\} \text{ for } x > \theta \text{ and zero elsewhere,} \tag{1.1}$$

where $-\infty < \theta < \infty$ and $\sigma > 0$.

We wish to estimate $\sigma$ by $\sigma_n = \hat{\sigma}$ where

$$\sigma_n = \sum_{i=1}^{n}(X_i - X_{1n})/(n-1) \text{ and } X_{1n} = \min(X_1, \ldots, X_n)\,. \tag{1.2}$$

From Epstein and Sobel ([5], Corollary 3) we have that

$$Y_n = 2(n-1)\sigma_n/\sigma \underset{=}{d} \chi^2_{2(n-1)} \tag{1.3}$$

where $\chi^2_k$ denotes a chi-square variable with $k$ degrees of freedom.

# 2 Fixed-width Confidence Interval Estimation of $\sigma$.

Let $I_n = (\sigma_n - d, \sigma_n + d)$ where $\hat{\sigma} = \sigma_n$ is given by (1.2). Define for $x > 0$

$$\psi(x) = (2\pi)^{-\frac{1}{2}}\int_{-x}^{x}\exp(-t^2/2)dt\,. \tag{2.1}$$

Set $\psi(z) = 1 - \alpha$ for $0 < \alpha < 1/2$ and let $\{z_n\}$ be an sequence of constants converging to $z$. In particular, if $z_n$ is the $(1-\alpha/2)^{th}$ fractile of the $t$-distribution with $n$ degrees of freedom, then

$$z_n = z\{1 + n^{-1}\Delta_0 + o(n^{-1})\} \text{ with } \Delta_0 = (1+z^2)/4$$

(see, for instance, Woodroofe [6], p. 993)). Now, for large $n$ (using the asymptotic normality of $(n-1)^{1/2}(\sigma_n - \sigma)$)

$$P(\sigma \in I_n) \geq 1 - \alpha \text{ implies that } \psi\left((n-1)^{1/2}d/\sigma\right) \geq \psi(z); \text{ or } n \geq [z^2\sigma^2/d^2] + 1\,, \tag{2.2}$$

where $[\cdot]$ denotes the largest integer contained in $(\cdot)$. Since $\sigma$ is unknown, then we resort to the following sequential rule:

$$\begin{aligned} R : N &= N(d) = t + 1 \text{ where for } m \geq 2 \\ t &= \inf\{n \geq m : n \geq z_n^2\sigma_n^2/d^2\}\,. \end{aligned} \tag{2.3}$$

After stopping at $N$, the confidence interval for $\sigma$ is given by

$$I_N = (\sigma_N - d, \sigma_N + d)\,. \tag{2.4}$$

The stopping rule (2.3) can be rewritten as

$$t = \inf\{n \geq m : S_n = \sum_{i=1}^{n} U_i \leq cn^\alpha L(n)\}\,,$$

where $U_i$ are i.i.d. as $\chi_2^2, c = 2d/\sigma z, \alpha = 3/2$ and

$$L(n) = 1 + (\frac{1}{2} - \Delta_0)/n + o(n^{-1})\,.$$

Now, we will state in Theorem 2.1 the general result of Woodroofe ([6], Theorem 2.4) which will be used in the sequal.

THEOREM 2.1 Let $F$ denote the distribution of $U_1$. Assume that

$$F(x) \leq Bx^a \text{ for all } x > 0$$

for some $B > 0$ and $a > 0$. (If the preceding condition is satisfied for all sufficiently small $x$, then it is satisfied for all $x$ with a possibly new $B$ but the same $a$). Let $E|X_1^r| < \infty$ for some $r > 2$. Also assume that $U_1$ has a density $f$ which is continuous $a.e.$ and that some power of the characteristic function of $U_1$ is integrable. If $r(2\alpha - 1) > 4$ and $ma > \beta$, then

$$E(t) = \lambda + \frac{\beta\nu}{\mu} - \beta L_0 - \frac{1}{2}\alpha\beta^2\tau^2\mu^{-2} + o(1)$$

as $c \to 0$ where

$$\begin{aligned} \nu &= \frac{\beta}{2\mu}[(\alpha-1)^2\mu^2 + \tau^2] - \sum_{n=1}^{\infty} n^{-1}E\{(S_n - n\alpha\mu)^+\} \\ \beta = (\alpha-1)^{-1}, \mu &= EU_1, \tau^2 = \operatorname{var} U_1 \text{ and } \lambda = (\mu/c)^\beta\,. \end{aligned}$$

Thus applying Theorem 2.1 with $\beta = (\alpha-1)^{-1} = 2, \mu = EU_i = 2, \tau^2 = \operatorname{var} U_i = 4, L_0 = \frac{1}{2} - \Delta_0, \lambda = \mu^\beta c^{-\beta}$, $\beta^2\tau^2\mu^{-2} = 4$ and $\nu = \frac{5}{2} - \sum_{n=1}^{\infty} n^{-1}E(S_n - 3n)^+$, we obtain

$$E(\tau) = (\sigma z/d)^2 + 2\Delta_0 - (3/2) - \sum_{n=1}^{\infty} n^{-1}E\{(S_n - 3n)^+\}\,. \tag{2.5}$$

Furthermore, if $z_n$ denotes the $(1-\alpha/2)^{th}$ fractile of the $t$-distribution with $n$ degrees of freedom, then (since $\Delta_0 = (1+z^2)/4$))

$$E(\tau) - (\sigma z/d)^2 = (z^2/2) - 1 - \sum_{n=1}^{\infty} n^{-1}E\left\{(S_n - 3n)^+\right\} + o(1) \text{ as } c \to 0\,. \tag{2.6}$$

Also from Woodroofe ([6], p. 986) we have, after specializing from gamma to $\chi_2^2$ density and performing linear interpolation in his Table 2.1, we obtain

$$\sum_{n=1}^{\infty} n^{-1} E(S_n - 3n)^+ = 1.438 .$$

So

$$E(t) = (\sigma z/d)^2 + (z^2/2) - 2.438 + o(1) \text{ as } c \to 0 . \tag{2.7}$$

# 3 Exact Expressions for the Expectation of the Stopping Time and the Confidence Coefficient.

In this section we derive the exact expressions for the stopping time and the confidence coefficient associted with the fixed-width confidence interval estimation and the stopping time associated with point estimation. Towards this, we need the following lemmas. Throughout this section we assume that $z_n \equiv z$.

Let $\lambda = (\sigma z/d)^2$ and $S^*_{i-1}$ be the sum of $i-1$ independent standard negative exponential random variables and let

$$b_{i-1} = \begin{cases} (i-1)i^{1/2}/\lambda^{1/2} & \text{for } i \geq m \\ 0 & \text{for } i \leq m-1 . \end{cases}$$

Then the joint density of $S^*_{m-1}, \ldots, S^*_{n-1}$ is

$$\{(m-2)!\}\, e^{-u_{n-1}} u_{m-1}^{m-2} , 0 < u_{m-1} < \ldots < u_{n-1} < \infty . \tag{3.1}$$

**Lemma 3.1**. Let $A_j(u) = \int_{b_{j-2}}^{u} A_{j-1}(v)dv$ for $j > m$, where $A_s(u) = u^{s-2}/(s-2)!$, for $2 \leq s \leq m$. Then

$$A_j(u) = \sum_{s=2}^{j-1} A_s(b_{j-2}) \frac{(u - b_{j-2})^{j-s}}{(j-s)!} , j > m . \tag{3.2}$$

By simple calculation the proof follows.

**Lemma 3.2**. We have, for $j > m$

$$P(t > j) = e^{-b_{j-1}} \left\{ \sum_{l=3}^{j} A_l(b_{j-1}) + 1 \right\} ,$$

with

$$P(t > m) = \int_{b_{m-1}}^{\infty} e^{-u} \frac{u^{m-2}}{(m-2)} du .$$

PROOF. $P(t > m) = \int_{b_{m-1}}^{\infty} e^{-u} \frac{u^{m-2}}{(m-2)} du$.

Consider

$$P(t > j) = P(S^*_{i-1} > b_{i-1}, i = m, \ldots, j) .$$

Since the $b_i$'s and the $S^*_i$ are increasing in $i$, by Lemma 1, we have

$$\begin{aligned} P(t > j) &= \int_{u_{j-1}=b_{j-1}}^{\infty} \cdots \int_{u_m=b_m}^{u_{m+1}} \int_{u_{m-1}=b_{m-1}}^{u_m} dP\left(S^*_{j-1} \in du_{j-1}, \ldots, S^*_{m-1} \in du_{m-1}\right) \\ &= \int_{u_{j-1}=b_{j-1}}^{\infty} \cdots \int_{u_{m-1}=b_{m-1}}^{u_m} e^{-u_{j-1}} \frac{u_{m-1}^{m-2}}{(m-2)!} du_{m-1} du_m \cdots du_{j-1} . \end{aligned}$$

If we define

$$A_j(u) = \int_{b_{j-2}}^{u} A_{j-1}(v)dv$$

then

$$\begin{aligned} P(t > j) &= \int_{b_{j-1}}^{\infty} e^{-u} A_j(u) du \\ &= e^{-b_{j-1}} A_j(b_{j-1}) + \int_{b_{j-1}}^{\infty} e^{-u} A_{j-1}(u) du \end{aligned}$$

after performing integration by parts once. By repeated integration by parts process we obtain the desired result.

REMARK 3.1. Notice that $P(t > i) = 1$ for $0 \leq i \leq m-1$ since $b_{i-1} = 0$ for $i = 1, 2, \ldots m-1$.

REMARK 3.2. We compute the $A_j$ recursively using Lemma 3.1 and compute $P(t > j)$ by using Lemma 3.2 and using the latter one can compute

$$E(t) = m + \sum_{j=m}^{\infty} P(t > j). \tag{3.3}$$

## An Exact Expression for the coverage Probability.

Here we derive an exact expression for the coverage probability. Towards this we need the following elementary result.

LEMMA 3.3. Let $a(c) = (\sqrt{\lambda} + c)^2$. Then

$$(n-1)(1 + c/\sqrt{\lambda}) \leq b_{n-2}$$

when $n \leq [n_2(c)]$ or when $n \geq [n_1(c)] + 1$ where

$$\begin{aligned} n_1(c) &= (1/2)\left\{(a+4) + a^{\frac{1}{2}}(a+4)^{\frac{1}{2}}\right\}, \\ n_2(c) &= (1/2)\left\{(a+4) - a^{\frac{1}{2}}(a+4)^{\frac{1}{2}}\right\}, \end{aligned}$$

and $[\cdot]$ denotes the largest integer contained in $(\cdot)$.

Furthermore,

(i). $n_2(c) < 1 + a$ if $a > 1/2, n_2(c) \geq 1 + a$ if $a \leq 1/2$, and

(ii). $n_1(c) > 1 + a$ for all $a$.

PROOF. The proof follows from solving the following equation

$$(n-1)\lambda(1 + c/\sqrt{\lambda})^2 = (n-2)^2.$$

In order to obtain (i) and (ii) solve the corresponding inequalities.

Let $b_{i-1} = (i-1)i^{1/2}/\lambda^{1/2}, i = m \cdots$. Then

$$\begin{aligned} \gamma &= \sum_{n=m}^{\infty} P(\sigma - d \leq \sigma_n \leq \sigma + d, t = n) \\ &= \sum_{n=m}^{\infty} P\left\{(n-1)\left(1 - z/\sqrt{\lambda}\right) \leq S^*_{n-1} \leq (n-1)\left(1 + z/\sqrt{\lambda}\right),\right. \\ &\qquad \left. S^*_{i-1} > b_{i-1}, m \leq i \leq n-1, S^*_{n-1} \leq b_{n-1}\right\}. \end{aligned} \tag{3.4}$$

In order to evaluate $\gamma$ we consider the following ranges for the summation variable $n$.

**Case 1.** Let $n$ be such that $n^{1/2}\lambda^{1/2} \leq (1 - z/\sqrt{\lambda})$, that is

$$n \leq \left(\sqrt{\lambda} - z\right)^2 \text{ or } n \leq \left[\left(\sqrt{\lambda} - z\right)^2\right].$$

In this range the probability of each summand is zero.

**Case 2.** $\left(1 - z/\sqrt{\lambda}\right) < n^{1/2}/\lambda^{1/2} \leq \left(1 + z/\sqrt{\lambda}\right)$. That is

$$\left(\sqrt{\lambda} - z\right)^2 < n \leq \left(\sqrt{\lambda} + z\right)^2 \text{ or } \left[\left(\sqrt{\lambda} - z\right)^2\right] + 1 \leq n \leq \left[\left(\sqrt{\lambda} + z\right)^2\right].$$

Let

$$I = (n-1)\left(1 - z/\sqrt{\lambda}\right) \leq S^*_{n-1} \leq (n-1)\left(1 + z/\sqrt{\lambda}\right), \; S^*_{i-1} > b_{i-1}, \; m \leq i \leq n-1, \; S^*_{n-1} \leq b_{n-1},$$

$$P\left(I, 1 - \frac{z}{\sqrt{\lambda}} < n^{1/2}\lambda^{1/2} < 1 + z/\sqrt{\lambda}\right) = P\left((n-1)\left(1 - z/\sqrt{\lambda}\right) \leq S^*_{n-1} \leq b_{n-1}, S^*_{i-1} > b_{i-1},\right.$$
$$\left. m \leq i \leq n-1\right).$$

**Case 3.** $n^{1/2}/\lambda^{1/2} > 1 + z/\sqrt{\lambda} \iff n > \left(\sqrt{\lambda}+z\right)^2$ or $n \geq \left[\left(\sqrt{\lambda}+z\right)^2\right] + 1$.

$$P\left(I, n^{1/2}\lambda^{1/2} > 1 + z/\sqrt{\lambda}\right) = P\left((n-1)\left(1 - z/\sqrt{\lambda}\right) \leq S^*_{n-1} \leq (n-1)\left(1+z/\sqrt{\lambda}\right),\right.$$
$$\left. S^*_{i-1} > b_{i-1}, m \leq i \leq n-1\right).$$

So

$$\gamma = \gamma_1 + \gamma_2 \tag{3.5}$$

where

$$\gamma_1 = \sum_{n=\max\{m,[(\sqrt{\lambda}-z)^2]+1\}}^{[(\sqrt{\lambda}+z)^2]} P\left((n-1)\left(1-z/\sqrt{\lambda}\right) \leq S^*_{n-1} \leq b_{n-1}\right.,$$
$$\left. S^*_{i-1} > b_{i-1}, m \leq i \leq n-1\right), \tag{3.6}$$

and

$$\gamma_2 = \sum_{n=1+[(\sqrt{\lambda}+z)^2]}^{\infty} P\left((n-1)\left(1-z/\sqrt{\lambda}\right) \leq S^*_{n-1}\right.$$
$$\left. \leq (n-1)\left(1+z/\sqrt{\lambda}\right), S^*_{i-1} > b_{i-1}, m \leq i \leq n-1\right). \tag{3.7}$$

Furthermore, we can write

$$\gamma_1 = \sum_{n=\max\{m,[(\sqrt{\lambda}-z)^2]+1\}}^{[(\sqrt{\lambda}+z)^2]} \left\{P\left(S^*_{n-1} > (n-1)\left(1-z/\sqrt{\lambda}\right),\right.\right.$$
$$\left.\left. S^*_{i-1} > b_{i-1}, m \leq i \leq n-1\right) - P(t > n)\right\}$$

and

$$\gamma_2 = \sum_{n=1+[(\sqrt{\lambda}+z)^2]}^{\infty} \left\{P(S^*_{n-1} \geq (n-1)\left(1-z/\sqrt{\lambda}\right), S^*_{i-1} > b_{i-1}, m \leq i \leq n-1\right.$$
$$\left. -P\left(S^*_{n-1} > (n-1)(1+z/\sqrt{\lambda}), S^*_{i-1} > b_{i-1}, m \leq i \leq n-1\right)\right\}.$$

So

$$\begin{aligned}
\gamma_1 + \gamma_2 &= \sum_{n=\max\{m,[(\sqrt{\lambda}-z)^2]+1\}}^{\infty} P\left(S^*_{n-1} > (n-1)\left(1-z/\sqrt{\lambda}\right), S^*_{i-1} > b_{i-1}, m \leq i \leq n-1\right) \\
&- \sum_{n=1+[(\sqrt{\lambda}+z)^2]}^{\infty} P\left(S^*_{n-1} > (n-1)\left(1+z/\sqrt{\lambda}\right), S^*_{i-1} > b_{i-1}, m \leq i \leq n-1\right) \\
&- \sum_{n=\max\{m,[(\sqrt{\lambda}-z)^2]+1\}}^{[(\sqrt{\lambda}+z)^2]} P(t > n) \\
&= T_1 - T_2 - T_3 \text{ (say)}.
\end{aligned} \tag{3.8}$$

If $a(-z) > 1/2$, then from Lemma 3.3 we have that $n_1(-z) > 1 + a(-z)$ and $n_2(-z) < 1 + a(-z)$. Hence we can write

$$\begin{aligned}
T_1 &= \sum_{n=\max\{m,\ 1+[a(-z)]\}}^{\infty} \\
&= \sum_{n=\max\{1+[a(-z)],\ m\}}^{[n_1(-z)]} + \sum_{n=[n_1(-z)]+1}^{\infty} \\
&= \sum_{n=\max\{1+[a(-z)],\ m\}}^{[n_1(-z)]} P\left(S^*_{n-1} > (n-1)\left(1-z/\sqrt{\lambda}\right), S^*_{i-1} > b_{i-1},\ m \leq i \leq n-1\right) \\
&+ \sum_{n=1+[n_1(-z)]}^{\infty} P(t > n-1).
\end{aligned} \tag{3.9}$$

Note that if $n_1(-z)] < \max\{1 + a(-z), m\}$ then the contribution from the first summation is zero, and the lower limit in the second summation should be $\max\{1 + a(-z), m\}$. This will be further elaborated under "special case". Also, since $1 + a(z) < n_1(z)$, we can write

$$\begin{aligned} T_2 &= \sum_{n=1+[(\sqrt{\lambda}+z)^2]}^{\infty} P\left(S^*_{n-1} > (n-1)\left(1 + z\sqrt{\lambda}\right), S^*_{i-1} > b_{i-1}, m \le i \le n-1\right) \\ &= \sum_{n=1+[(\sqrt{\lambda}+z)^2]}^{[n_1(z)]} + \sum_{n=1+[n_1(z)]}^{\infty} \\ &= \sum_{n=1+[(\sqrt{\lambda}+z)^2]}^{[n_1(z)]} P\left(S^*_{n-1} > (n-1)\left(1 + z/\sqrt{\lambda}\right),\ S^*_{i-1} > b_{i-1},\ i \le n-1\right) \\ &\quad + \sum_{n=1+[n_1(z)]}^{\infty} P(t > n-1). \end{aligned} \tag{3.10}$$

Thus

$$\begin{aligned} \gamma &= \sum_{n=\max\{m,1+[a(-z)]\}}^{[n_1(-z)]} P\left(S^*_{n-1} > (n-1)\left(1 - z/\sqrt{\lambda}\right), S^*_{i-1} > b_{i-1}, m \le i \le n-1\right) \\ &\quad - \sum_{n=1+[a(z)]}^{[n_1(z)]} P\left(S^*_{n-1} > (n-1)\left(1 + z/\sqrt{\lambda}\right), S^*_{i-1} > b_{i-1}, m \le i \le n-1\right) \\ &\quad + \sum_{n=[n_1(-z)]}^{[n_1(z)]-1} P(t > n) - \sum_{n=\max\{m,1+[a(-z)]\}}^{[a(z)]} P(t > n). \end{aligned} \tag{3.11}$$

**Special Case.** If $[n_1(-z)] < \max\{m, 1 + [a(-z)]\}$, then one can write

$$T_1 = \sum_{n=\max\{m-1,[a(-z)]\}}^{\infty} P(t > n).$$

Hence

$$\begin{aligned} \gamma &= -\sum_{n=1+[a(z)]}^{[n_1(z)]} P\left(S^*_{n-1} > (n-1)\left(1 + z/\sqrt{\lambda}\right), S^*_{i-1} > b_{i-1}, m \le i \le n-1\right) \\ &\quad + \sum_{n=\max\{m-1,[a(-z)]\}}^{[n_1(z)]-1} P(t > n) - \sum_{n=\max\{m,1+[a(-z)]\}}^{[a(z)]} P(t > n). \end{aligned} \tag{3.12}$$

Again, the last two terms will simplify to

$$P(t > \max\{m-1, [a(-z)]\}) + \sum_{n=1+[a(z)]}^{[n_1(z)]-1} P(t > n). \tag{3.13}$$

Also, as noted in the proof of Lemma 3.2, since the $b_i$'s and the $S^*_i$ are increasing in $i$, we have

$$\begin{aligned} &P\left(S^*_{n-1} > (n-1)\left(1 \pm z/\sqrt{\lambda}\right), S^*_{i-1} > b_{i-1},\ m \le i \le n-1\right) \\ &= \int_{u_{n-1}=(n-1)(1\pm z/\sqrt{\lambda})}^{\infty} e^{-u_{n-1}} \int_{b_{n-2}}^{u_m} \cdots \int_{b_{m-1}}^{u_m} \frac{u_{m-1}^{m-2}}{(m-2)!} du_{m-1} \cdots du_{n-1} \\ &\qquad = \int_{u_{n-1}=(n-1)(1\pm z/\sqrt{\lambda})}^{\infty} e^{-u_{n-1}} A_n(u_{n-1}) du_{n-1} \\ &= e^{-B_{n-1}} \left\{ \sum_{l=3}^{n} A_l(B_{n-1}) + 1 \right\}, \text{ where } B_{n-1} = (n-1)\left(1 \pm z/\sqrt{\lambda}\right), \end{aligned} \tag{3.14}$$

after performing integration by parts repeatedly. If $B_{n-1} < b_{n-2}$, then

$$P\left(S^*_{n-1} > B_{n-1}, S^*_{i-1} > b_{i-1}, m \le i \le n-1\right) = P\left(S^*_{i-1} > b_{i-1}, m \le i \le n-1\right) = P(t > n-1).$$

**Remark 3.3.** For numerical computations, we define

$$\bar{A}_k(b_{j-1}) = A_k(b_{j-1})e^{-b_{j-1}}$$

and

$$D_k(j-1) = A_k(B_{j-1})e^{-B_{j-1}}$$

where

$$B_{j-1} = (j-1)\left(1 \pm z/\sqrt{\lambda}\right),$$

Table 3.1: Exact Values of $Et$ and the Confidence Coefficient for Various Values of $\lambda$ and $m$.

| $z$ | $\lambda$ | $\lambda^{1/2}$ | $\sigma/d$ | $m=4$ $Et$ | $m=4$ $ECC^*$ | $m=8$ $Et$ | $m=8$ $ECC$ | $m=10$ $Et$ | $m=10$ $ECC$ |
|---|---|---|---|---|---|---|---|---|---|
| | 3.84 | 1.96 | 1.0 | 5.21 | 1.000 | 9.23 | 1.000 | 10.09 | 1.000 |
| | 8.64 | 2.94 | 1.5 | 7.97 | 0.999 | 14.30 | 1.000 | 11.47 | 0.999 |
| 1.96 | 15.37 | 3.92 | 2.0 | 12.64 | 0.778 | 23.75 | 0.933 | 15.44 | 0.997 |
| | 24.01 | 4.90 | 2.5 | 19.58 | 0.741 | 37.65 | 0.897 | 22.16 | 0.897 |
| | 34.57 | 5.88 | 3.0 | 29.00 | 0.766 | 55.77 | 0.922 | 31.58 | 0.847 |
| | 47.06 | 6.86 | 3.5 | 40.92 | 0.802 | 77.66 | 0.946 | 43.57 | 0.860 |
| | 6.63 | 2.58 | 1 | 6.73 | 1.000 | 8.27 | 1.000 | 10.69 | 1.000 |
| | 14.92 | 3.86 | 1.5 | 12.31 | 0.920 | 10.23 | 0.999 | 15.13 | 1.000 |
| | 16.52 | 5.15 | 2.0 | 21.74 | 0.810 | 14.62 | 0.997 | 24.31 | 1.000 |
| 2.575 | 41.44 | 6.44 | 2.5 | 35.48 | 0.836 | 21.56 | 0.843 | 38.10 | 0.911 |
| | 59.68 | 7.72 | 3.0 | 53.44 | 0.880 | 31.09 | 0.830 | 56.13 | 0.929 |
| | 81.22 | 9.01 | 3.5 | 75.27 | 0.914 | 43.15 | 0.850 | 77.96 | 0.950 |

$^*ECC$ = Exact Confidence Coefficient

and compute $\bar{A}_k(b_{j-1})$ and $D_k(j-1)$ recursively after rewriting (3.2) and (3.15) in terms of $\bar{A}_k(b_{j-1})$ and $D_k(j-1)$, and hence evaluate $P(t > j)$ and the probabilities

$$P(S^*_{j-1} > B_{j-1}, S^*_{i-1} > b_{i-1}, m \le i \le j-1).$$

**Remark 3.4.** In the sequential rule, we can replace $z$ by a sequence $\{z_n\}$ converging to $z$. For instance $z_n$ could be the $(1-\alpha/2)^{th}$ quantiles of Student's t-distribution with $n$ degrees of freedom. In the latter case

$$z_i = z\left\{1 + (1+z^2)(4i)^{-1} + o(i^{-1})\right\}.$$

Then $b_{i-1}$ will be an increasing sequence provided $i \ge 2$. This is satisfied because we can always choose $m \ge 2$ or 3 (see the definition of $b_{i-1}$).

The first order asymptotic value of $Et$ is $\lambda$ and the second order asymptotic value for $Et$ (using Theorem 2.1) is given by

$$Et - \lambda = -1.50 - 1.438 + o(1) = -2.988 + o(1).$$

From Table 3.1, we infer that the asymptotic values for $Et$ are close to the true values when $\sigma/d \ge 1.5$. The surprise is that the exact confidence coefficient decreases with $\sigma/d$ for a while and increases from there on, but still falling short of the nominal confidence coefficient. When $\sigma/d = 1$, the actual confidence coefficient exceeds the nominal confidence coefficient. It seems one should take at lease 10 for $m$ in order for the exact confidence coefficient to be reasonably close to the nominal value.

# 4 Point Estimation of $\sigma$.

Let the loss incurred in estimating $\sigma$ by $\sigma_n$ where $\sigma_n$ is given by (1.2) be given by

$$L_n = (\sigma_n - \sigma)^2 + cn.$$

Then

$$EL_n = \sigma^2(n-1)^{-1} + cn = \beta_n(c) \text{ (say)}. \tag{4.1}$$

Setting $\partial\beta_n(x)/\partial n = 0$, we obtain

$$n = \sigma/c^{1/2} + 1\,. \tag{4.2}$$

Since $\sigma$ is unknown, we resort to the following sequential rule. The stopping time $N = t + 1$ where for $m \geq 2$

$$\begin{aligned} t &= \inf\left\{n \geq m : n \geq \sigma_n/c^{1/2}\right\} \\ &= \inf\left\{n \geq m : S^*_{n-1} \leq n(n-1)/\gamma\right\}, \gamma = \sigma/c^{1/2}, \end{aligned} \tag{4.3}$$

where $\gamma$ is the optimal fixed-sample size required when $\sigma$ is known and $S^*_{n-1}$ is the sum of $n-1$ standard exponential random variables. Thus

$$P(t > j) = P(S^*_{i-1} > i(i-1)/\gamma, i = m, \ldots, j) \tag{4.4}$$

and from Remark 3.2 we have

$$Et = m + \sum_{j=m}^{\infty} P(t > j)\,. \tag{4.5}$$

Hence, one can readily evaluate $Et$ for various values of $\gamma$ after evaluating $P(t > j)$ using Lemma 3.2 with $b_{i-1} = i(i-1)/\gamma$. These are tabulated in Table 4.1 for some value of $m$.

**Table 4.1:** $Et$ for Some Selected Values of $m$.

| $\gamma^{-1}$ | | 1.0 | 0.5 | 0.1 | 0.05 | 0.01 |
|---|---|---|---|---|---|---|
| $Et$ | $m = 4$ | 4.00 | 4.07 | 9.35 | 19.02 | 99.66 |
| | $m = 8$ | 8.00 | 8.00 | 10.42 | 19.42 | 99.69 |
| | $m = 10$ | 10.00 | 10.00 | 11.26 | 19.55 | 99.69 |

Towards the second order asymptotic results, from Govindarajulu and Sarkar [2] we have

$$Et = \gamma + 0.374 - 1 + o(1) = \gamma - 0.626 + o(1)\,. \tag{4.6}$$

From Table 4.1 we infer that the asymptotic values for $Et$ are very close to the exact values for $\gamma \geq 10$.

**ACKNOWLEDGEMENT**. The author expresses his sincere thanks to Helena Trusczynska for her assistance in computing the numerical values in Tables 3.1 and 4.1 and the referee for his helpful comments.

# REFERENCES

[1] Starr, N. and Woodroofe, M. Further remarks on sequential point estimation: The exponential case. *Ann. Math. Statist.* **43** (1972) 1147-1154.

[2] Govindarajulu, Z. and Sarkar, S. Sequential estimation of the scale parameter in exponential distribution with unknown location. *Utilitas Mathematica* **40** (1991) 161-178.

[3] Govindarajulu, Z. Fixed-width confidence interval estimation of scale parameter of exponential distribution with unknown location. *Journal of the Orissa math. Soc.* **4** (2) (1985) 77-83.

[4] Mukhopadhyay, N. Minimum risk point estimation of the mean of a negative exponential distribution. *Sankhyā* Ser. A **49** (1) (1987) 105-112.

[5] Epstein, B. and Sobel, M. Some theorems relevant to life testing from an exponential distribution. *Ann. Math. Statist.* **25** (1954) 373-381.

[6] Woodroofe, M. Second order approximation for sequential point and interval estimation. *Ann. Statist.* **5** (1977) 984-995.