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SOME PROPERTIES OF STARLIKE FUNCTIONS WITH RESPECT TO SYMMETRIC-CONJUGATE POINTS

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ABSTRACT. Let A be the class of all analytic functions in the unit disk U such that $f(0) = f'(0) - 1 = 0$. A function $f \in A$ is called starlike with respect to $2n$ symmetric-conjugate points if $Re\ zf'(z)/f_n(z) > 0$ for $z \in U$, where

$$f_n(z) = \frac{1}{2n}\sum_{k=0}^{n-1}[\omega^{-k}f(\omega^k z) + \omega^k\overline{f(\omega^k \bar{z})}],$$

$\omega = exp(2\pi i/n]$. This class is denoted by S_n^* and was studied in [1]. A sufficient condition for starlikeness with respect to symmetric-conjugate points is obtained. In addition, images of some subclasses of S_n^* under the integral operator $I : A \to A, I(f) = F$ where

$$F(z) = \frac{c+1}{(g(z))^c}\int_0^z f(t)(g(t))^{c-1}g'(t)dt, \quad c > 0$$

and $g \in A$ is given are determined.

KEY WORDS AND PHRASES: symmetric-conjugate points; starlike; differential subordinations; integral operator; strongly starlike; α-convex.

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1. INTRODUCTION

Let $m \geq 1$ be an integer and let A_m be the class of all functions f that are analytic in the unit disk U and having the power series expansion of the form

$$f(z) = z + a_{m+1}z^{m+1} + a_{m+2}z^{m+2} + \ldots, \quad z \in U.$$

We set $A \equiv A_1$.

In [1] the concept of starlike functions with respect to $2n$ symmetric-conjugate points was introduced. We recall that for a positive integer n and for $\omega = exp(2\pi i/n)$, a function $f \in A$ is called a starlike function with respect to $2n$ symmetric-conjugate points if

$$Re zf'(z)/f_n(z) > 0, \quad z \in U,$$

where

$$f_n(z) = \frac{1}{2n}\sum_{k=0}^{n-1}[\omega^{-k}f(\omega^k z) + \omega^k\overline{f(\omega^k\bar{z})}]. \tag{1.1}$$

The class of all such function is denoted by S_n^*. Note that $S_n^* \subseteq C$, where C is the class of close-to-convex functions.

The following relations can be deduced from (1.1).

$$f_n' = \frac{1}{2n}\sum_{k=0}^{n-1}[f'(\omega^k z) + \overline{f'(\omega^k\bar{z})}], \tag{1.2}$$

$$f''(z) = \frac{1}{2n}\sum_{k=0}^{n-1}[\omega^k f''(\omega^k z) + \omega^{-k}\overline{f''(\omega^k\bar{z})}], \tag{1.3}$$

$$\begin{gathered} f_n(\omega^j z) = \omega^j f_n(z), \quad f_n(\bar{z}) = \overline{f_n(\bar{z})}, \\ f_n'(\omega^j z) = f_n'(z), \quad f_n'(\bar{z}) = \overline{f_n'(z)}. \end{gathered} \tag{1.4}$$

In this paper we shall determine a sufficient condition for starlikeness with respect to symmetric-conjugate points. In addition, we find the images of certain subclasses of S_n^* under the integral operator $I : A \to A, I(f) = F$ where,

$$F(z) = \frac{c+1}{(g(z))^c}\int_0^z f(t)(g(t))^{c-1}g'(t)dt, \tag{1.5}$$

$c \geq 0$ and $g \in A$ is a given function. The case $g(z) \equiv z$ was discussed in [1]. A more general integral operator was studied in [2].

2. PRELIMINARIES

In order to prove our main results, we need the following definitions and lemmas. Let us first recall the definition of subordination. If $f, g \in A$ and g is univalent, f is subordinate to g, written $f \prec g$, or $f(z) \prec g(z)$, if $f(0) = g(0)$ and $f(U) \subset g(U)$. Also, a function $f \in A$ is called strongly starlike of order $\alpha, \alpha \in (0, 1]$ if

$$zf'(z)/f(z) \prec ((1+z)/(1-z))^\alpha$$

The class of all such functions is denoted by $S^*(\alpha)$. A function $f \in A$ is called α -convex, $\alpha \in R$ if

$$Re[(1-\alpha)zf'(z)/f(z) + \alpha((zf''(z)/f'(z)) + 1)] > 0,$$

$z \in U$. The class of all such functions is denoted by M_α.

LEMMA 2.1. *[4] Let $m \geq 1$ be an integer and*

$$p(z) = 1 + p_m z^m + p_{m+1} z^{m+1} + \ldots, \quad z \in U, \tag{2.1}$$

be analytic in U. If the function p is not with positive real part in U, then there is a point $z_0 \in U$ such that $p(z_0) = is$, $z_0 p'(z_0) = t$, where s, t are real and $t \leq -m(1+s^2)/2$.

LEMMA 2.2. *If $f \in A_m$ satisfies*

$$|f''(z)/f'(z)| \leq 1 + m/2, \quad z \in U,$$

then for all $z \in U$

i) $Re\ f(z)/(zf'(z)) > 1/2$,

ii) $|(zf'(z)/f(z)) - 1| < 1$

PROOF. It is clear that (i) and (ii) are equivalent. Let $p(z) = 2f(z)/(zf'(z)) - 1$. Then p has the form (2.1) and

$$zf''(z)/f'(z) = (1 - p(z) - zp'(z))/(p(z) + 1).$$

Suppose p is not with a positive real part. Then by Lemma 2.1 there is a $z_0 \in U$ such that $p(z_0) = is$, $z_0 p'(z_0) = t$, where $t \leq -m(1+s^2)/2$. Consequently,

$$\begin{aligned} |z_0 f''(z_0)/f'(z_0)|^2 &= ((1-t)^2 + s^2)/(1+s^2) \\ &\geq [(1 + m(1+s^2)/2)^2 + s^2]/(1+s^2) \\ &\geq (1 + m/2)^2, \end{aligned}$$

which contradicts the hypothesis of this lemma. The proof is now complete. The case $m = 1$ of Lemma 2.2 can be found in [5].

LEMMA 2.3. *[2] Let $\alpha \in (0, 1]$. For $c = 0$ suppose that $g \in S^\star(1-\alpha)$, while $g \in M_{1/c}$, for $c > 0$. If the function $f \in A$ satisfies*

$$g(z)f'(z)/(g'(z)f(z)) \prec ((1+z)/(1-z))^\alpha$$

then the function F defined by (1.5) is also in A, $F(z)/z \neq 0$ for $z \in U$ and

$$g(z)F'(z)/(g'(z)F(z)) \prec ((1+z)/(1-z))^\alpha.$$

LEMMA 2.4. *[3] Let $P(z)$ be analytic function in U with $Re\ P(z) > 0$, $z \in U$, and let h be a convex function in U. If p is analytic in U with $p(0) = h(0)$, then*

$$p(z) + P(z)zp'(z) \prec h(z) \quad \textit{implies} \quad p(z) \prec h(z).$$

3. MAIN RESULTS.

THEOREM 3.1. *Let $f \in A_m$, $m \geq 2$, and let n be a positive integer. If*

$$|f''(z)/f_n'(z)| \leq (m^2 - 1)/(4m), \tag{3.1}$$

$z \in U$, where $f_n(z)$ is defined by (1.2), then $f \in S_n^\star$

PROOF. From (1.4) and (3.1) we deduce

$$|\omega^k f''(\omega^k z)/f_n'(z)| \le (m^2-1)/(4m),$$

and

$$|\omega^{-k}\overline{f''(\omega^k \bar{z})}/f_n'(z)| \le (m^2-1)/(4m).$$

Combining these relations with (1 3) to get

$$|f_n''(z)/f_n'(z)| \le (m^2-1)/(4m), \quad z \in U.$$

Since $(m^2-1)/(4m) \le 1 + m/2$, then Lemma 2.2 can be applied to f_n to deduce, in particular, $f_n(z)/z \neq 0$ for $z \in U$. To complete the proof, let $p(z) = zf'(z)/f_n(z)$, then we need to show that $Re\ p(z) > 0$. Note that since f and f_n are in A_m, so p has the form (2.1) for $m \ge 1$. In addition

$$zf''(z)/f_n'(z) = (f_n(z)/(zf_n'(z)))(zp'(z) + p(z)(zf_n'(z)/f_n(z) - 1)).$$

Assume p is not with positive real part in U. Then by Lemma 1.1, there is a point $z_0 \in U$ such that $p(z_0) = is$, $z_0p'(z_0) = t$ and $t \le -m(1+s^2)/2$. Using the conclusions of Lemma 2.2 for f_n, we obtain

$$\begin{aligned} |z_0 f''(z_0)/f_n'(z_0)| &\ge 1/2|t + is(z_0 f_n'(z_0)/f_n(z_0) - 1)| \\ &\ge 1/2(|t| - |s|) \\ &\ge 1/2(m(1+s^2)/2 - |s|) \\ &\ge (m^2-1)/(4m), \end{aligned}$$

which contradicts the hypothesis (3.1). Hence $f \in S_n^\star$. This completes the proof of this theorem.

THEOREM 3.2. *Suppose $\alpha \in (0,1]$, $c \ge 0$ and $n \ge 1$ is an integer. Let $g \in S^\star(1-\alpha)$ be a function with the power series expansion*

$$g(z) = z + g_1 z^{n+1} + g_2 z^{2n+1} + \dots,$$

$z \in U$, where all the coefficients g_j are real. In addition, suppose that $g \in M_{1/c}$ for $c > 0$. Consider the integral operator $I : A \to A, I(f) = F$, where F is given by (1.5). If

$$g(z)f'(z)/(g'(z)f_n(z)) \prec ((1+z)/(1-z))^\alpha, \tag{3.2}$$

then

$$g(z)F'(z)/(g'(z)F_n(z)) \prec ((1+z)/(1-z))^\alpha,$$

where f_n and F_n are the functions associated with f and F as given in (1.1), respectively.

PROOF. First, we show that $F_n = I(f_n)$. Using (1.5) one can easily write $F(z)$ in the following form:

$$F(z) = \frac{c+1}{(g(z)/z)^c} \int_0^1 f(xz)(g(xz)/(xz))^{c-1} g'(xz) x^{c-1} dx.$$

From the expansion form of $g(z)$, it follows that

$$\frac{1}{2n}\omega^{-k} F(\omega^k z) = \frac{c+1}{(g(z)/z)^c} \int_0^1 \frac{1}{2n}\omega^{-k} f(\omega^k xz)(g(xz)/xz)^{c-1} g'(xz) x^{c-1} dx,$$

and

$$\frac{1}{2n}\omega^k \overline{F(\omega^k \bar{z})} = \frac{c+1}{(g(z)/z)^c} \int_0^1 \frac{1}{2n}\omega^k \overline{f(\omega^k x\bar{z})}(g(xz)/(xz))^{c-1} g'(xz)^{c-1} x dx.$$

Now by summation and (1.1) we deduce easily that $F_n = I(f_n)$. Replacing z by $\omega^k z$ and then by $\omega^k \bar{z}$, $k = \{0, 1, ..., n-1\}$ in (3.2) and using the relations (1.2) and (1.4) and also the fact that

$$g(\omega^k z) = \omega^k g(z), \quad g(\omega^k \bar{z}) = \omega^k \overline{g(z)}, \quad g'(\omega^k z) = g'(z), \quad g'(\omega^k \bar{z}) = \overline{g'(z)}.$$

We deduce the relation

$$g(z)f_n'(z)/(g'(z)f_n(z)) \prec ((1+z)/(1-z))^\alpha.$$

Applying Lemma 2.3 to the above to get

$$arg(G(z)zF_n'(z)/F_n(z) + c) < \alpha\pi/2, \tag{3.3}$$

where

$$G(z) = g(z)/(zg'(z)).$$

Let

$$P(z) = G(z)(G(z)zF_n'(z)/F_n(z) + c)^{-1}. \tag{3.4}$$

From (3.3) and the fact that $g \in S^*(1-\alpha)$, we easily deduce from (3.4) that

$$Re\ P(z) > 0.$$

Let

$$p(z) = g(z)F'(z)/(g'(z)F_n(z)).$$

Lemma 2.3 shows that $p(z)$ is analytic in U. Hence multiplication of (1.5) by g^c and differentiating the new equation we obtain

$$G(z)zF'(z) + cF(z) = (c+1)f(z) \tag{3.5}$$

and

$$G(z)zF_n'(z) + cF_n(z) = (c+1)f_n(z). \tag{3.6}$$

Substituting in (3 5)

$$G(z)F'(z) = p(z)F_n(z)$$

then differentiating the new equation and using (3.6) to get

$$\begin{aligned} p(z) + P(z)zp'(z) &= g(z)f'(z)/(g'(z)f_n(z)) \\ &\prec ((1+z)/(1-z))^{\alpha}, \end{aligned} \tag{3.7}$$

where $P(z)$ is given by (3.4) with $Re\ P(z) > 0$. Applying Lemma 2.4 to (3.7) to deduce

$$Re\ p(z) = Re\ g(z)F'(z)/(g'(z)F_n(z)) > 0.$$

This completes the proof of this theorem.

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