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SURFACE TEMPERATURE DETERMINATION FROM BOREHOLE MEASUREMENTS: REGULARIZATION AND ERROR ESTIMATES

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ABSTRACT. The authors consider the problem of determining the temperature distribution $u(x,t)$ on the half-line $x = 0, t > 0$, from measurements at an interior point, for all $t > 0$. As is well-known, this is an ill-posed problem Using the Tikhonov method, the authors give a regularized solution, and assuming the (unknown) exact solution is in $H^{\alpha}(\mathbb{R}), \alpha > 0$ They give an error estimate of the order $1/(1n1/\varepsilon)^{\alpha}$ for $\varepsilon \to 0$, where $\varepsilon > 0$ is a bound on the measurement error

KEY WORDS AND PHRASES. Surface temperature distribution, Tikhonov regularization, error estimate.
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1. INTRODUCTION.

We consider the problem of determining the temperature distribution $u(x,t)$ on the half line $x = 0, t > 0$ from measurements at $x = 1$ for all $t > 0$ The problem is of importance in Geophysics. It has been considered by various authors (see references) Of particular relevance to the present work are the papers of Anderssen and Saull [1] and of Anderssen [2], and the paper of Carasso [3]. The paper of Anderssen and the one of Anderssen and Saull. give the physical background and concentrate on numerical and stabilization problems. The work of Carasso makes use of the Tikhonov regularization method. The difference between Carasso's result and ours can best be seen in Remark 2 at the end of this paper. The paper of Talenti and Vessella [4] is also of interest, it gives stability estimates between the two solutions We should note that the problem under consideration is ill-posed in the sense that a solution does not always exist, and that whenever they exist, solutions do not depend continuously on the given (measured) data. The ill-posedness is discussed by various authors (see [1], [3] and [5]). We shall regularize the problem using the Tikhonov method. We shall give error estimates, assuming the exact solution can be extended to a function in $H^{\alpha}(\mathbb{R})$, $\alpha > 0$.

2. MAIN RESULTS.

We propose to solve the heat equation

$$u_t - u_{xx} = 0 \qquad 0 < x < \infty, \qquad t > 0, \tag{2.1}$$

subject to the following conditions

$$u(x,0) = f(x), \qquad u(1,t) = g(t), \tag{2 2}$$

and a decay condition, which, following [5], we take as

$$\lim_{x \to \infty} x^{-2} \ln \left(\int_0^T u^2 u^2(x,t)dt \right) = 0 \qquad \text{for any } T > 0 \tag{2.3}$$

If we write $v(t)$ for $u(0,t)$, then we have the following representation formula, valid for $u(x,0) = 0$, which, without loss of generality, we assume here

$$u(x,t) = \int_0^t v(s)K(x,t-s)ds \tag{2.4}$$

where (see Widder [6] and Cannon [7])

$$K(x,t) = (4\pi)^{-1}\, xt^{-3/2}\, exp(-x^2/4t) \qquad \text{if } t > 0$$
$$= 0 \qquad \text{if } t \leq 0 \tag{2.5}$$

To find $v(t)$, it is therefore sufficient to solve the following integral equation

$$(4\pi)^{-1} \int_0^t v(s)(t-s)^{-3/2} exp(-1/4(t-s)) = g(t), \quad t > 0 \tag{2.6}$$

which is conveniently written as

$$\frac{1}{\sqrt{2\pi}} \int_0^t v(s)K(t-s) = h(t), \quad t > 0, \tag{2.7}$$

where

$$h(t) = 2\sqrt{2\pi}\, g(t)$$
$$K(t) = t^{-3/2} exp(-1/4t), \quad t > 0$$
$$= 0, \qquad t \leq 0$$

We assume $g(t)$ to be an L_2-function, and, hence, $h(t)$ is an L_2-function. Taking the Fourier transform of both sides of (2.7) gives

$$\widehat{K}(p)\widehat{v}(p) = \widehat{h}(p) \tag{2.8}$$

where

$$\widehat{K}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K(t)e^{-ipt}\, dt \tag{2.9}$$

and similarly for h and v, which are understood to vanish for $t < 0$. We shall consider the following regularized equation

$$\beta\,\widehat{v}_\beta + |\,\widehat{K}\,|^2\,\widehat{v}_\beta = \overline{\widehat{K}}\,\widehat{h}, \quad \beta > 0, \tag{2.10}$$

Note that a solution $\widehat{v}_\beta$ of (2.10) always exists and is Lipschitzian in $\widehat{h}$ The inverse Fourier transform v_β of $\widehat{v}_\beta$, which we shall take as our regularized solution for an appropriate β, is therefore Lipschitzian in h since the L_2-Fourier transform is a unitary operator on $L_2(\mathbb{R})$.

Now, let v be the exact solution of (2.7) for $h = h_0$, and suppose the error between the measured right hand side of (2.7) and the (unknown) exact value is less than $\varepsilon > 0$, i.e.,

$$|\,h - h_0\,|_2 < \varepsilon \text{ where } |\cdot|_2 = L_2\text{-norm} \tag{2.11}$$

Then, we have the following error estimate, taking $\beta = \varepsilon$

THEOREM 2.1. Let (2.11) hold, and suppose v is in $H^\alpha(\mathbb{R})$, $\alpha > 0$ Then

$$|\,v_\varepsilon - v\,|_2 < c/(1n \cdot 1/\varepsilon)^{c\alpha} \tag{2.12}$$

where

$$v_\varepsilon(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{v}_\varepsilon(p)\, exp(ipt)dp$$

and $c = 2^{1/2}B$ with B defined in (2 20)

PROOF. By (2 8) and (2 9), we have for $\beta > 0$

$$\beta(\widehat{v}_\beta - \widehat{v}) + |\widehat{K}|^2(\widehat{v}_\beta - \widehat{v}) = -\beta\widehat{v} + \overline{\widehat{K}}(\widehat{h} - \widehat{h}_0) \tag{2.13}$$

Taking the L_2-inner product with $\widehat{v}_\beta - \widehat{v}$ and estimating, we have the inequality

$$\beta\,|\widehat{v}_\beta - \widehat{v}|_2^2 + |\widehat{K}(\widehat{v}_\beta - \widehat{v})|_2^2 \le \beta\,|\widehat{v}|_2\,|\widehat{v}_\beta - \widehat{v}|_2 + |\widehat{h} - \widehat{h}_0|_2\,|\widehat{K}(\widehat{v}_\beta - \widehat{v})|_2$$

$$\le \beta\,|\widehat{v}|_2^2 + \tfrac{1}{2}\beta\,|\widehat{v}_\beta - \widehat{v}|_2^2 + \tfrac{1}{2}\left(|\widehat{h} - \widehat{h}_0|_2^2 + |\widehat{K}(\widehat{v}_\beta - \widehat{v})|_2^2\right) \tag{2.14}$$

Letting $\beta = \varepsilon < 1$, and noting that

$$\left|\widehat{h} - \widehat{h}_0\right|_2 = |h - h_0|_2 < \varepsilon$$

we then have, after some rearrangements

$$\varepsilon\,|\widehat{v}_\varepsilon - \widehat{v}|_2^2 + \left|\widehat{K}(\widehat{v}_\varepsilon - \widehat{v})\right|_2^2 \le \varepsilon\left(|\widehat{v}|_2^2 + 1\right) \tag{2.15}$$

In particular,

$$\varepsilon\,|\widehat{v}_\varepsilon - \widehat{v}|_2^2 \le \varepsilon\left(|\widehat{v}|_2^2 + 1\right) \tag{2.16}$$

and

$$\left|\widehat{K}(\widehat{v}_\varepsilon - \widehat{v})\right|_2^2 \le \varepsilon\left(|\widehat{v}|_2^2 + 1\right) \tag{2.17}$$

Now, taking the L_2-inner product of both sides of (2.13) with $|p|^{2\alpha}(\widehat{v}_\varepsilon - \widehat{v})$ and estimating, we have (for $\beta = \varepsilon$) the inequality

$$\varepsilon\,\left||p|^\alpha(\widehat{v}_\varepsilon - \widehat{v})\right|_2^2 + \left||p|^\alpha\,\widehat{K}(\widehat{v}_\varepsilon - \widehat{v})\right|_2^2$$

$$\le \varepsilon\,\left||p|^\alpha\widehat{v}\right|_2\,\left||p|^\alpha(\widehat{v}_\varepsilon - \widehat{v})\right|_2 + \left||p|^\alpha\widehat{K}\right|_\infty\left|\widehat{h} - \widehat{h}_0\right|_2\,\left||p|^\alpha(\widehat{v}_\varepsilon - \widehat{v})\right|_2 \tag{2.18}$$

In view of the following result (see [8])

$$\left|\widehat{K}(p)\right| = 2\,exp\left(-\left(|p|/2\right)^{1/2}\right)$$

we have

$$\left||p|^\alpha\widehat{K}\right|_\infty < \infty.$$

From (2.18), we have, in particular

$$\varepsilon\,\left||p|^\alpha(\widehat{v}_\varepsilon - \widehat{v})\right|_2^2 \le \varepsilon\,\left||p|^\alpha(\widehat{v}_\varepsilon - \widehat{v})\right|_2\left(\left||p|^\alpha\widehat{K}\right|_\infty + \left||p|^\alpha\widehat{v}\right|_2\right)$$

i.e.,

$$\left||p|^\alpha(\widehat{v}_\varepsilon - \widehat{v})\right|_2 \le \left||p|^\alpha\widehat{K}\right|_\infty + \|v\|_\alpha \tag{2.19}$$

where

$$\|v\|_\infty \equiv \left||p|^\alpha\widehat{v}\right|_2 \qquad \text{(the } H^\alpha\text{-norm)}.$$

Let

$$B^2 = 2\left(\left||p|^\alpha\widehat{K}\right|_\infty^2 2\,\|v\|_\alpha^2\right) + 1 \tag{2.20}$$

Then, we have

$$|v_\varepsilon - v|_2 + \|v_\varepsilon - v\|_\alpha \le 2B$$

The foregoing step establishes an H^α-bound on $v_\varepsilon - v$. As a final step in our proof, we estimate the smallness of $|v_\varepsilon - v|_2$ for $\varepsilon \to 0$. To this end, we have for any $a > 0$

$$\int_{|p|\le a} |\widehat{v}_\varepsilon - \widehat{v}(p)|^2\,dp \le \int_{|p|\le a} |\widehat{K}(p)|^2/\,\widehat{K}(a)|^2|(\widehat{v}_\varepsilon - \widehat{v})(p)|^2\,dp$$

$$\le exp(a/2)^{1/2}\,B^2\varepsilon \tag{2.21}$$

On the other hand

$$\int_{|p|\geq a} |(\widehat{v}_\varepsilon - \widehat{v})(p)|^2\, dp \leq a^{-2\alpha} \int_{-\infty}^{\infty} |\,|p|^\alpha (\widehat{v}_\varepsilon - \widehat{v})|^2\, dp \leq B^2 / a^{2\alpha}$$

Now, let $a = p_\varepsilon$ be the positive solution of the equation

$$|\, p \,|^{2\alpha}\, exp\,(\,|\, p \,|\,/2)^{1/2} = 1/\varepsilon \tag{2.22}$$

Then

$$|\widehat{v}_\varepsilon - \widehat{v}|_2^2 \leq 2B^2/p_\varepsilon^{2\alpha} \tag{2.23}$$

Taking the logarithm of each side of (2 22) gives

$$2\alpha\, lnp_\varepsilon + (p_\varepsilon/2)^{1/2} = ln\,(1/\varepsilon) \tag{2.24}$$

Since $p_\varepsilon \to \infty$ for $\varepsilon \to 0$, it follows that, for all $\varepsilon > 0$ sufficiently small, the left-hand side of (2.24) is less than $p_\varepsilon^{1/2}$ or equivalently

$$\left(1/p_\varepsilon^2\right) < 1/(ln1/\varepsilon)^4 \tag{2.25}$$

Hence, for all small $\varepsilon > 0$, we have from (2.23)

$$|v_\varepsilon - v|_2^2 = |\widehat{v}_\varepsilon - \widehat{v}|_2^2 = c^2/(ln1/\varepsilon)^{4\alpha}, \quad c^2 = 2B^2 \tag{2.26}$$

This completes the proof.

REMARK 1. Among all choices of $\beta = \varepsilon^\delta$, $\delta > 0$, as Tikhonov parameters, our choice $\beta = \varepsilon$, in a sense, is, from the point of view of our analysis, the optimal one, except for the coefficient c in the inequality (2.12). First, if $\delta > 1$, then our analysis breaks down, since the steps leading to (2.19) are no longer valid. If $0 < \delta < 1$, then, retracing the steps from (2.13) to (2.24), we would have, for $\beta = \varepsilon^\delta$, an estimate of the form

$$|v_\beta - v|_2 \leq c'(\,\|\, v \,\|_{\alpha}, \varepsilon)/(\delta ln1/\varepsilon)^{2\alpha} \tag{2.27}$$

where c' is bounded as $\varepsilon \to 0$. We omit the details there.

REMARK 2. In [3], Carasso (under weaker conditions on the exact solution) produced a regularized solution that approximates the exact solution only at interior points $x > 0$.

In closing, we would like to point out that multidimensional analogues of our problem are much more complicated. They are considered, e.g., in [9], where the problem is treated numerically, and in [10], where the problem is formulated as a moment problem.

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