SOME REMARKS ABOUT MACKEY CONVERGENCE

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ABSTRACT. In this paper, we examine Mackey convergence with respect to K-convergence and bornological (Hausdorff locally convex) spaces. In particular, we prove that: Mackey convergence and local completeness imply property K; there are spaces having K- convergent sequences that are not Mackey convergent; there exists a space satisfying the Mackey convergence condition, is barrelled, but is not bornological; and if a space satisfies the Mackey convergence condition and every sequentially continuous seminorm is continuous, then the space is bornological.

1. INTRODUCTION.

This note contains a somewhat miscellaneous collection of results that do or do not hold for a space ("space" here means: Hausdorff locally convex space) that satisfies the Mackey convergence condition; specifically, we consider the following properties of a space (X, τ):

(a) (X, τ) satisfies the <u>Mackey convergence condition</u> [1; 5.1.29, page 158]: For every null sequence *(x(n))* there is there is a closed, bounded absolutely convex set (i.e., a *disk*) B, such that *x(n)* $\to 0$ in X_B, where X_B is the linear span of B equipped with the topology of the gauge of B. Equivalently, [1; 5.1.3, page 151] for every null sequence *(x(n))* there is a sequence *(a(n))* of positive numbers such that $a(n) \to \infty$ as $n \to \infty$, and $a(n)x(n) \to 0$ in X.

(b) (X, τ) satisfies the <u>Katowice property</u> (also <u>property K</u> or is a <u>K- space</u>) [1;1.2.15, page 20], [2; page 19]: For every null sequence *(x(n))* there is a subsequence *(y(n))* of *(x(n))* such that $\sum\limits_{n\,=1}^{\infty} y(n)$ converges in X.

(c) (X, τ) is bornological.

Spaces that satisfy (a) have been studied recently with regard to topics such as inductive limits. For example, the reader may peruse [3] and the references therein. Property (b) was originally defined by Mazur and Orlicz (see [1; 1.4, page 30]) and was used in [4] to find some sequential conditions under which a space is bornological.

We first show that Mackey convergence and local completeness imply property K, where we recall that a space (X, τ) is <u>locally complete</u> [1; 5.1.6, page 152] if for each closed bounded disk B, the space X_B is a Banach space. However, we also prove that there are spaces having K- convergent sequences that are not Mackey convergent and that a weak Mackey convergent sequence is a Mackey convergent sequence for the original topology. Hence, a space that satisfies property (a) with respect to its weak topology also satisfies *Shur's property*- every weakly convergent sequence is convergent for the original topology.

Antosik and the first author ask in [4] whether a barrelled space for which every sequentially continuous seminorm is continuous must be bornological. Bonet [5] proves the answer is "no". We prove here that if the barrelled assumption is replaced by (a), then the answer is "yes". This slightly improves the last result of [4].

Finally, we exhibit a space that satisfies (a), is barrelled, but is not bornological. This example is related to an old problem of Grothendieck (see [6] and [7]): *Is there a quasibarrelled (DF) -space satisfying (a) but not bornological?*

2. MAIN RESULTS.

THEOREM 1. *Let* (X, τ) *be a space.*

(i) If (X, τ) *is locally complete and satisfies the Mackey convergence condition, then it satisfies property K.*

(ii) Any weak Mackey convergent sequence is Mackey convergent with respect to the original topology.

(iii) If (X, τ) *satisfies the Mackey convergence condition with respect to its weak topology, then every weakly convergent sequence is originally convergent in X.*

PROOF. *(i).* Suppose *(x(n))* is any null sequence in (X, τ). Local completeness and the Mackey convergence condition together imply that *(x(n))* is a null sequence in X_B for some closed bounded disk B, and X_B is a Banach space. As a Banach space satisfies property K, *(x(n))* has a subsequence that is series convergent in X_B to some $x \in X_B$. Because B is closed and the identity map from X_B into X is continuous, this subsequence is series convergent in X.

(ii). Now suppose *(x(n))* is a weak Mackey convergent sequence in X. Then there is a sequence *(a(n))* of positive numbers tending to infinity, and such that $a(n)x(n) \to 0$ weakly. The set $\{a(n)x(n) : n \in N\}$ is even τ- bounded in X. Therefore, it suffices to find a sequence *($\lambda(n)$)* of positive numbers such that $\lambda(n) \to 0$ but $a(n)\lambda(n) \uparrow \infty$.

(iii). Obvious consequence of (ii). ◆

The authors do not know if property K implies the Mackey convergence condition in general, but the next example shows that one can have a sequence that satisfies property K but is not Mackey convergent.

EXAMPLE 1. Let $X = l^\infty$, equipped with the following seminorms: For each $x = (x(n)) \in X$, define:

(a) For each $k \in N$, $p_k(x) = |x(k)|$.

(b) For each $y \in l^1$, $p_y(x) = \sum\limits_{k=1}^{\infty} y(k)x(k)$.

We let τ be the locally convex topology generated by these seminorms. Note that by our definition of τ, a sequence *(x(n))* is τ-convergent to $x \in X$ if and only if:

(i) $x(n) \to x$ coordinatewise; namely, $x(n,k) \to x(k)$, as $n \to \infty$, and

(ii) There is an $M > 0$ such that for every $k, n \in N$, $|x(n,k)| \le M$.

Define *(z(n))* $\subset X$ by $z(n) = e(n)$, where $e(n) = (0,0,...,0,1,0,...)$ (1 in the *n*th place). Clearly, $z(n) \to 0$ and is even summable, hence *(z(n))* is a K-sequence. However, because of (i) and (ii), there cannot be a sequence *(a(n))* of positive numbers tending to infinity such that $a(n)z(n) \to 0$. ◆

We note here that the above space does not satisfy property K; consider the sequence $z(n) = (0,0,...,0,1,1,...)$ (0 in the first *n*-1 places, then 1's). *(z(n))* is a null sequence in X, but has no summable subsequence.

THEOREM 2. *If (X, τ) satisfies the Mackey convergence condition and every sequentially continuous seminorm on X is continuous, then (X, τ) is bornological.*

PROOF. Assume that p is a seminorm on X such that if $x(n) \to 0$ then

(p(x(n))) is bounded. Then there is a sequence of positive scalars $\alpha(n) \to \infty$ such that $\alpha(n)x(n) \to 0$, which in turn means that for some $M > 0$, $p(\alpha(n)x(n)) \le M$ for each positive integer *n*. Hence, $p(x(n)) \to 0$. This shows that *p* is sequentially continuous, and by assumption, *p* is continuous. It follows that (X, τ) is bornological.   ◆

We now present an example of a space that satisfies the Mackey convergence condition, is barrelled, but is not bornological. However, we will also show that the space does not have a fundamental sequence of bounded sets so is not a solution to the Grothendieck problem.

EXAMPLE 2. Let *X* be the set of all real-valued functions on R that are constant except on a countable set. Equip *X* with the locally convex topology τ generated by the family of seminorms $P = \{ p_t : t \in R\}$, where

$$p_t(x) = |x(t)|, \qquad x \in X.$$

Claim 1 : (X, τ) satisfies the Mackey convergence condition. To see this, suppose *(x(n))* is any null sequence in (X, τ). For each $n \in N$, let T_n denote the set on which *(x(n))* is not constant. We count T_n by $T_n = \{ t(1,n), t(2,n), \ldots\}$. Denoting by 1 the characteristic function, we may represent $x(n) \in X$ by

$$x(n, t) = c(n) + \sum_{k=1}^{\infty} \lambda(n,k) 1\{(n,k)\},$$

where c(n) and λ(*n, k*) are constants, for $n, k \in N$. It is clear from this that *(c(n))* is a null sequence of real numbers. Next, denote by *T* the countable set $\cup\{T_n : n \in N\}$, and label it by $T = \{t(1), t(2), \ldots\}$. Because $x(n, t(k)) \to 0$ as $n \to \infty$, there is a sequence *(a(n))* ↑ ∞ such that both $a(n)x(n, t(k)) \to 0$ and $a(n)c(n) \to 0$, as $n \to \infty$. Thus, *(x(n))* is a Mackey convergent sequence.

Claim 2 : (X, τ) is barrelled. If $f \in X'$ (*X'* denotes the continuous dual of *X*), then *f* is of the form

$$f(x) = \alpha(1)x(t(1)) + \ldots + \alpha(k)x(t(k)),$$

for $\{ t(1), \ldots, t(k) \} \subset R$, and some nonzero scalars $\alpha(1), \ldots, \alpha(k)$. To prove that *X* is barrelled, it is enough to show equicontinuity for sequences of

pointwise bounded members of X'. With this in mind, let $(f(n)) = (f(n,x))$ be any sequence such that for every $x \in X$ and every $n \in N$, $|f(n,x)|$ is bounded. We will find a $C > 0$ and a $p \in P$ such that $|f(n,x)| \leq Cp(x), n \in N, x \in X$.

For each $n \in N$, let $S_n = \{ t(1,n), t(2,n), \ldots t(k_n,n) \}$, for which

$$f(n, x) = \sum_{i=1}^{k_n} \alpha(i, n) x(t(k_i, n)),$$

$\alpha(i,n) \neq 0$ constants. Put

$$S = \cup \{ S_n : n \in N \}.$$

S is finite; for if not, then there would exist for each $n \in N$, $t(n) \in S_{\beta_n} \setminus [\cup\{ S_k : k = 1, \ldots, \beta_{n-1} \}]$. For brevity, relable β_n by n, and then denote by y an element of X that takes the values given by $y(t(k_n,n)) = [\alpha(k_n,n)]^{-1}n$. Then $f(n,y) = nk_n \to \infty$ as $n \to \infty$, and this is bad because then $(f(n))$ is not pointwise bounded at y.

Hence, we are allowed to write $S = \{ t(1), t(2), \ldots t(K) \}$ for some $K \in N$, and

$$f(n, x) = \sum_{i=1}^{K} \alpha(i,n) x(t_i).$$

Put $\alpha(i) = max\{|\alpha(i,n)|: n \in N\}$, for $i = 1, \ldots, K$. Each $\alpha(i) < \infty$ because of the pointwise boundedness of the sequence $(f(n))$. It follows that with $C = max\{\alpha(i,) : i = 1, \ldots, K\}$, we have

$$| f(n,x)| \leq \sum_{i=1}^{K} \alpha(i) |x(t_i)|.$$

$$\leq Cp(x),$$

where p is the seminorm corresponding to the set S. This means that $(f(n))$ is equicontinuous.

Claim 3: (X, τ) is not bornological. To see this, consider the linear functional f on X given by $f(x) = c$, where $x(t) = c$, except on a countable set. f maps bounded sets in X to bounded sets in R, but is not continuous; hence by [1; 6.1.18, page 168], X cannot be bornological.

Claim 4 : (X, τ) does not have a fundamental sequence of bounded sets (f.s.b.).
To prove this, suppose for a contradiction that (X, τ) has a f.s.b., (B_n). Then in particular, for each $n = 1, 2, ...,$ we can find $a_n > 0$ such that for each $x \in B_n$, $|x(n)| \le a_n$. Let us define $A \subset X$ by $A = \{y_m : m \in N\}$, where we define each y_m as follows: put $y_m(t) = 0$ if $t \ne m$, and $y_m(m) = 1 + a_m$. From this we see that for any $t \in R$, we have that if $t = m_0$ for some $m_0 \in N$, then $sup\{| y_m(t) | : m \in N\}$ is $1 + a_{m_0}$; otherwise $sup\{| y_m(t) | : m \in N\} = 0$. Therefore, A is τ- bounded. On the other hand, it is easy to see that A is not contained in any B_n.   ◆

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