Internat. J. Math. & Math. Sci.
VOL. 18 NO. 4 (1995) 681-688

A NOTE ON KÖTHE-TOEPLITZ DUALS OF CERTAIN SEQUENCE SPACES AND THEIR MATRIX TRANSFORMATIONS

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(In the memory of Late Professor B. Kuttner)

(Received November 5, 1992 and in revised form September 22, 1993)

ABSTRACT. In this paper we define the sequence spaces $S\ell_\infty(p)$, $Sc(p)$ and $Sc_0(p)$ and determine the Köthe-Toeplitz duals of $S\ell_\infty(p)$. We also obtain necessary and sufficient conditions for a matrix A to map $S\ell_\infty(p)$ to ℓ_∞ and investigate some related problems.

KEY WORDS AND PHRASES. Sequence spaces, Köthe-Toeplitz duals, Matrix transformations.
1992 AMS SUBJECT CLASSIFICATION CODES. 40H05

1. INTRODUCTION.

If $\{p_k\}$ is a sequence of strictly positive real numbers, then

$$\ell_\infty(p) = \{ x: \sup_k | x_k |^{p_k} < \infty\};$$

$$c(p) = \{ x : | x_k - \ell |^{p_k} \to 0 \text{ for some } \ell \};$$

$$c_0(p) = \{ x : | x_k |^{p_k} \to 0\} .$$

For detailed discussion on these spaces we refer [1,4,5,6,7,8].

Recently Kizmaz [3] defined the following sequence spaces:

If $\Delta x = (x_k - x_{k+1})$, then

$$\ell_\infty(\Delta) = \{ x = \{ x_k\} : \Delta x \in \ell_\infty \};$$

$$c(\Delta) = \{ x = \{x_k\} : \Delta x \in c \};$$

$$c_0(\Delta) = \{ x = \{x_k\} : \Delta x \in c_0 \}.$$

These spaces are Banach spaces with norm

$$\| x \|_\Delta = | x_1| + \| \Delta x \|_\infty$$

Furthermore, since $\ell_\infty(\Delta)$ is a Banach space with continuous co-ordinates (that is, $\|x^n-x \|_\Delta \to 0$ implies $| x_k^n - x_k | \to 0$ for each $k \in N$, as $n \to \infty$), it is a BK-space.

If X is a sequene space, we define [2]

$$X^{\alpha} = \{ a = (a_k) : \sum_{k=1}^{\infty} | a_k x_k | < \infty \text{ for each } x \in X \};$$

$$X^{\beta} = \{ a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \text{ is convergent for each } x \in X \};$$

X^{α} and X^{β} are called the α-(or Köthe-Toeplitz) and β-(or generalized Köthe-Toeplitz), dual spaces of X respectively.

We now define some new sequence spaces. If $\Delta x = x_k - x_{k-1}$, we define

$$S\ell_{\infty}(p) = \{ x = \{x_k\} : \Delta x \in \ell_{\infty}(p) \};$$

$$Sc(p) = \{ x = \{x_k\} : \Delta x \in c(p) \};$$

$$Sc_o(p) = \{ x = \{x_k\} : \Delta x \in c_o(p) \}.$$

We observe that if $x_k = k$(for all $k \in \mathbb{N}$) then $x \in S\ell_{\infty}(p)$ but $x \notin \ell_{\infty}(p)$.

PROPERTIES : (i) $S\ell_{\infty}(p)$ and $Sc(p)$ are paranormed spaces with the paranorm $g(x) = \sup_k | \Delta x_k |^{p_k/M}$ where $M = \max(1, \sup p_k)$ if and only if $0 < \inf p_k \le \sup p_k < \infty$.

(ii) If $p = \{p_k\}$ is a bounded sequence, then $Sc_o(p)$ is a paranormed space with the paranorm

$$g(x) = \sup_k |\Delta x_k|^{p_k/M}$$

The proof of these properties are similar to the proof given in [6, Th.1].

2. DUALS

THEOREM 1. Let $p_k > 0$ for every k. Then

$$(S\ell_{\infty}(p))^{\alpha} = \bigcap_{N=1}^{\infty} \left\{ y = \{ y_n \} : \sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{n} N^{1/p_m} \right\} | y_n | < \infty \right\}$$

PROOF. We need to prove that $(S\ell_{\infty}(p))^{\alpha}$ is the set of all sequences y such that, for every positive integer N,

$$\sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{n} N^{1/p_m} \right\} | y_n | < \infty.$$

If $x \in S\ell_{\infty}(p)$, then by definition, $|\Delta X_n|^{p_n}$ is bounded, so that, for some N, $|\Delta X_n|^{p_n} \le N$; thus $|\Delta X_n| \le N^{1/p_n}$. So

$$|x_n| \le \sum_{m=1}^{n} N^{1/p_m} ; \tag{2.1}$$

$$(\text{by the relation } x_n = \sum_{v=1}^{n} \Delta x_v)$$

Thus, if

$$\sum_{n=1}^{\infty}\left\{\sum_{m=1}^{n} N^{1/p_m}\right\} |y_n| < \infty \tag{2.2}$$

holds, then

$$\sum_{n=1}^{\infty} |x_n y_n| < \infty$$

Hence, (2.2) is a sufficient condition for $y \in (S\ell_\infty(p))^\alpha$.

Conversely, if N is given we can define $x \in S\ell_\infty(p)$ by $x_m = \sum_{n=1}^{m} N^{1/p_n}$,

so that(2.2) is necessary for y to be in $(S\ell_\infty(p))^\alpha$.

Now we raise the following question :

Is it true that $(S\ell_\infty(p))^\alpha$ is the set of sequences y such that, for every positive integer N,

$$\sum_{n=1}^{\infty} n N^{1/p_n} \; | y_n| < \infty \; ? \tag{2.3}$$

In other words, is it true that

$$(S\ell_\infty(p))^\alpha = \bigcap_{N=1}^{\infty}\left\{ y = (y_n) : \sum_{n=1}^{\infty} n N^{1/p_n} \; | y_n| < \infty \right\}?$$

It does not follow at once from Theorem 1 that this conjecture is false, since it is not obvious that the assertion that (2.2) holds for all N is not equivalent to the assertion that (2.3) holds for all N. Indeed, there are some sequences $\{p_n\}$ for which these assertions are equivalent. However, for general $\{p_n\}$ they need not be equivalent. We give examples to show that

(A) It is possible to choose $\{p_n\}$ such that there is a $y = \{y_n\}$ for which (2.3) holds for all N, but (2.2) does not. Thus (2.3) is not always sufficient.

(B) It is possible to choose $\{p_n\}$ such that there is a y for which (2.2) holds for all n, but (2.3) does not. Thus (2.3) is not always necessary.

EXAMPLE 1. Take

$$\left\{ \begin{array}{l} p_{2k-1} = 1 \\ p_{2k} = 1/k \end{array} \right\} k = (1,2,3,\ldots)$$

Then take

$$\left\{ \begin{array}{l} y_{2k-1} = \dfrac{1}{k^3} \\ y_{2k} = 0 \end{array} \right\} (k = 1,2,3,\ldots)$$

Since $y_{2k} = 0$, it is only the odd terms which contribute to (2.2) or (2.3). For these terms we take $p_n = 1$ and thus the sum on the left of (2.3) is

$$N \sum_{k=1}^{\infty} \frac{2k-1}{k^3} < \infty$$

But for $n = 2k\text{-}1$, $k \geq 2$,

$$\sum_{m=1}^{n} N^{1/p_m} \geq N^{1/p_{2k-2}} = N^{k-1}.$$

Thus the sum on the left of (2.2) is greater than or equal to

$$\sum_{k=2}^{\infty} \frac{N^{k-1}}{k^3} = \infty \text{ if } N > 1.$$

EXAMPLE 2. Take

$$p_n = \begin{cases} \dfrac{1}{\log r} & (n = 2^r, r=2,3,4,\ldots) \\ 1 & (\text{otherwise}) \end{cases}$$

Then take

$$y_n = \begin{cases} \dfrac{1}{2^r r^2} & (n = 2^r, r=2,3,4,\ldots) \\ 0 & (\text{otherwise}) \end{cases}$$

In the sums (2.2), (2.3) all the terms vanish except for $n = 2^r$, $r = 2,3,4,\ldots$ So we need consider only those terms. If $n = 2^r$, $r \geq 2$ then there are 2^r-(r-1) terms in the sum

$$\sum_{m=1}^{n} N^{1/p_m} \text{ for which } p_m = 1 \text{ so that}$$

$$\sum_{m=1}^{n} N^1/p_m = (2^r-(r-1))N + \sum_{\rho=2}^{r} N^{\log \rho}$$

But $N^{\log \rho} = \rho^{\log N}$ so that for fixed N

$$\sum_{\rho=2}^{r} N^{\log\rho} = \sum_{\rho=2}^{r} \rho^{\log N} = O\,(r^{\log N+1}) = o\,(2^r).$$

Thus, for fixed N, and $n = 2^r$ we have

$$\sum_{m=1}^{n} N^{1/p_m} = O\,(2^r)$$

so that

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} N^{1/p_m}\right) |\, y_n \,| = O\left(\sum_{r=2}^{\infty} \frac{1}{r^2}\right) < \infty$$

But

$$\sum_{n=1}^{\infty} nN^{1/p_n} |\, y_n \,| = \sum_{r=2}^{\infty} 2^r\, N^{\log r} \frac{1}{2^r\, r^2}$$

$$= \sum_{r=2}^{\infty} \frac{r^{\log N}}{r^2} \text{ (since } N^{\log r} = r^{\log N})$$

$$= \infty \text{ if } \log N \geq 1,$$

i.e. for $N = 3, 4, 5, \ldots$.

We now consider the second dual of $(S\ell_\infty(p))$ i.e. $(S\ell_\infty(p))^{\alpha\alpha}$.

Is it true that

$$(S\ell_\infty(p))^{\alpha\alpha} = \bigcup_{N=1}^{\infty} \left\{ Z: \sup_n \frac{|z_n|}{\sum_{m=1}^{n} N^{1/p_m}} < \infty \right\} ?$$

In other words, is it true that $(S\ell_\infty(P))^{\alpha\alpha}$ is the set of sequences $z = \{z_n\}$ which are such that, for some N

$$z_n = O\left(\sum_{m=1}^{n} N^{1/p_m}\right) ? \tag{2.4}$$

In order to see that this conjecture is true, we shall first prove a lemma.

LEMMA 1. Suppose that, for each N, $\{a_n^{(N)}\}$ is a sequence of positive numbers, and that, for fixed n, $a_n^{(N)}$ is non-decreasing in N. Let X denote the set of sequences $\{y_n\}$ which are such that, for all N,

$$\sum_{n=1}^{\infty} a_n^{(N)} |y_n| < \infty \tag{2.5}$$

Then X^α is the set of all $\{z_n\}$ such that, for some N

$$z_n = O(a_n^{(N)}) \tag{2.6}$$

PROOF. The result that (2.6) is sufficient for $z \in X^\alpha$ is trivial; for, if (2.6) holds for some N then since (2.5) holds for all N it holds for that particular N, whence

$$\sum_{n=1}^{\infty} |y_n z_n| < \infty$$

The result that (2.6) is necessary is not so obvious. Suppose it is false that there is some N for which (2.6) holds.

Then, for every N,

$$\frac{z_n}{a_n^{(N)}} \text{ is unbounded.}$$

Hence, we can determine an increasing sequence $\{n_N\}$ of positive integers such that

$$\frac{|z_{n_N}|}{a_{n_N}^{(N)}} \geq N^2$$

Now define $y = \{y_n\}$ by

$$y_n = \begin{cases} \dfrac{1}{N^2 a_{n_N}^{(N)}} & (n = n_N,\ N = 1,2,3,\ldots) \\ 0 & \text{otherwise} \end{cases}$$

Now given any fixed N we have for all $M \geq N$

$$y_{n_M} = \frac{1}{M^2 a_{n_M}^{(M)}} \leq \frac{1}{M^2 a_{n_M}^{(N)}}$$

(since $a_n^{(N)}$ is non-decreasing for fixed n).

The terms in (2.5) for which n is not equal to n_M for some M are 0; hence the contribution to (2.5) of these terms with $n \geq n_N$ is less than or equal to

$$\sum_{M=N}^{\infty} \frac{1}{M^2} < \infty$$

Since there are only a finite number of terms with $n < n_N$ the series (2.6) converges. This holds for every N; hence $y \in X$.

But when $n = n_N$ we have $|y_n z_n| \geq 1$. Hence $\sum_{n=1}^{\infty} |y_n z_n|$ diverges so that $z \notin X^\alpha$

The conjecture preceding Lemma 1 now follows from the result for $(S\ell_\infty(p))^\alpha$ by taking

$$X = (S\ell_\infty(p))^\alpha, \quad a_n^{(N)} = \sum_{m=1}^{n} N^{1/p_m}$$

3. MATRIX TRANSFORMATIONS

In this section we find necessary and sufficient conditions for $A \in (S\ell_\infty(p), \ell_\infty)$. We need the following lemma.

LEMMA 2. Let $p_k > 0$ for every k. Then

$$(S\ell_\infty(p))^\beta = \bigcap_{N=2}^{\infty} \left\{ a = \{a_k\}: \sum_{k=1}^{\infty} a_k \sum_{m=1}^{k} N^{1/p_m} \text{ converges}, \sum_{k=1}^{\infty} |R_k| N^{1/p_k} < \infty \right\}$$

Where $R_k = \sum_{v=k}^{\infty} a_v$.

PROOF. Suppose that $x \in S\ell_\infty(p)$. Then there is an integer

$$N > \max\left(1, \sup_k |\Delta x_k|^{p_k}\right) \text{ such that}$$

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} R_k \Delta x_k - R_{n+1} \sum_{k=1}^{n} \Delta x_k \tag{3.1}$$

where $n \in \mathbb{N}$.

Since

$$\sum_{k=1}^{\infty} |R_k| |\Delta x_k| \leq \sum_{k=1}^{\infty} |R_k| N^{1/p_k} < \infty,$$

it follows that

$$\sum_{k=1}^{\infty} R_k \Delta x_k \text{ is absolutely convergent.}$$

Also, by Corollary 2 [3], the convergence of

$$\sum_{k=1}^{\infty} a_k \left(\sum_{m=1}^{k} N^{1/p_m}\right) \text{ implies that } \lim_{n\to\infty} R_{n+1} \sum_{m=1}^{k} N^{1/p_m} = 0$$

Hence, it follows from (3.1) that

$$\sum_{k=1}^{\infty} a_k x_k \text{ is convergent for each } x \in S\ell_\infty(p).$$

This gives $a \in (S\ell_\infty(p))^\beta$.

Conversely, suppose that $a \in (S\ell_\infty(p))^\beta$, then by definition, $\sum_{k=1}^{\infty} a_k x_k$ is convergent for each $x \in S\ell_\infty(p)$.

Since $e = (1,1,1,\ldots) \in S\ell_\infty(p)$ and $x = \left\{\sum_{m=1}^{k} N^{1/p_m}\right\} \in S\ell_\infty(p)$, then $\sum_{v=1}^{\infty} a_v$ and $\sum_{v=1}^{\infty} a_v \left(\sum_{m=1}^{v} N^{1/p_m}\right)$ are convergent. By using Corollary 2 [3] we find that

$$\lim_{n\to\infty} R_{n+1} \sum_{m=1}^{k} N^{1/p_m} = 0.$$

Thus, we obtain from (3.1) that the series $\sum_{k=1}^{\infty} R_k \, \Delta x_k$ converges for each $x \in S\ell_\infty(p)$.

Note that $x \in S\ell_\infty(p)$ if and only if $\Delta x \in \ell_\infty(p)$. This implies that $R = \{R_k\} \in (\ell_\infty(p))^\beta$. It now follows from Theorem 2 [4] that

$$\sum_{k=1}^{\infty} | R_k | N^{1/p_k} \text{ converges for all } N > 1.$$

We now find necessary and sufficient conditions for a matrix A to map $S\ell_\infty(p)$ to ℓ_∞.

THEOREM 2. Let $p_k > 0$ for every k. Then $A \in (S\ell_\infty(p), \ell_\infty)$ if and only if

(i) $$\sup_n \left| \sum_{k=1}^{\infty} a_{nk} \left(\sum_{m=1}^{k} N^{1/p_m}\right) \right| < \infty, \ N > 1;$$

(ii) $$\sup_n \left[\sum_{k=1}^{\infty} N^{1/p_k} \left| \sum_{v=k}^{\infty} a_{nv} \right| \right] < \infty, \ N > 1.$$

PROOF. We first prove that these conditions are necessary.

Suppose that $A \in (S\ell_\infty(p), \ell_\infty)$. Since $x = \left(\sum_{m=1}^{k} N^{1/p_m}\right)$ belongs to $S\ell_\infty(p)$, the condition (i) holds. In order to see that (ii) is necessary we assume that for $N > 1$,

$$\sup_n \left[\sum_{k=1}^{\infty} N^{1/p_k} \left|\sum_{v=k}^{\infty} a_{nv}\right| \right] = \infty.$$

Let the matrix B be defined by $B = (b_{nk}) = \left(\sum_{v=k}^{\infty} a_{nv}\right)$.

Then it follows from Theorem 3 [4] that $B \notin (\ell_\infty(p), \ell_\infty)$. Hence, there is a sequence $x \in \ell_\infty(p)$ such that $\sup_k |X_k|^{p_k} = 1$ and $\sum_{k=1}^{\infty} b_{nk} x_k \neq O(1)$.

We now define the sequence y by

$$y_k = \sum_{v=1}^{k} x_v (k \in N),\ y_o = 0. \text{ Then } y \in S\ell_\infty(p)$$

$$\text{and} \quad \sum_{k=1}^{\infty} a_{nk} y_k = \sum_{k=1}^{\infty} b_{nk} x_k \neq O(1).$$

This contradicts that $A \in (S\ell_\infty(p), \ell_\infty)$. Thus, (ii) is necessary.

We now prove the sufficiency part of the theorem. Suppose that (i) and (ii) of the theorem hold. Then $A_n \in (S\ell_\infty(p))^\beta$ for each $n \in N$.

Hence $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$ converges for each $n \in N$ and for each $x \in S\ell_\infty(p)$. Following the argument used in Lemma 2, we find that if $x \in S\ell_\infty(p)$ such that $\sup_k |\Delta x_k|^{p_k} < N$, then

$$\left| \sum_{k=1}^{\infty} a_{nk} x_k \right| \leq \sum_{k=1}^{\infty} N^{1/p_k} \left| \sum_{v=k}^{\infty} a_{nv} \right|$$

$$\leq \sup_n \left[\sum_{k=1}^{\infty} N^{1/p_k} \left| \sum_{v=k}^{\infty} a_{nv} \right| \right] < \infty$$

This proves that $Ax \in \ell_\infty$. Hence, the theorem is proved.

ACKNOWLEDGEMENT. The authors thank the refree for helpful suggestions.

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