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COMPUTATION OF GREEN'S MATRICES FOR BOUNDARY VALUE PROBLEMS ASSOCIATED WITH A PAIR OF MIXED LINEAR REGULAR ORDINARY DIFFERENTIAL OPERATORS

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ABSTRACT. An algorithm for the computation of Green's matrices for boundary value problems associated with a pair of mixed linear regular ordinary differential operators is presented and two examples from the studies of acoustic waveguides in ocean and transverse vibrations in nonhomogeneous strings are discussed.

KEY WORDS AND PHRASES: Nonexplicitly mixed, matchingly mixed, boundary value problem, Green's matrices.
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1. INTRODUCTION

Recently, a new class of problems of the type where different differential operators are defined over two adjacent intervals, involving certain mixed (interface) conditions are studied in [1,2,3,4]. These problems involve a pair of differential operators of the type $\tau_1 u_1 = \sum_{k=0}^{n} P_k D^k u_1 = \lambda u_1$, defined on the interval $J_1 = [a,b]$ and $\tau_2 u_2 = \sum_{k=0}^{m} Q_k D^k u_2 = \lambda u_2$ defined on the adjacent interval $J_2 = [b,c]$, $-\infty < a < b < +\infty$, where λ is an unknown constant (eigenvalue) and the functions u_1 and u_2 are required to satisfy certain mixed conditions at the interface $x = b$. In most of the cases, the complete set of physical conditions on the system gives rise to selfadjoint eigenvalue problems associated with the pair (τ_1, τ_2). Based on the interface conditions these problems can be classified into three types, namely (i) where the values of u_1 and u_2 are not explicitly related to each other at $x = b$, (ii) where u_1 and u_2 are required to satisfy the continuity conditions at $x = b$, and (iii) where u_1 and u_2 satisfy certain matching conditions at $x = b$.

The methods presented in [4] for the construction of Green's matrices for the boundary value problems (BVPs) associated with (τ_1, τ_2) are theoretical in nature and involve lengthy calculations. Here, in this paper we present (i) simpler algorithms for the computation of Green's matrices for the BVPs associated with (τ_1, τ_2), and (ii) construct the Green's matrices for the problems found in some physical situations.

Before indicating the division of the work into sections, we introduce a few notations and make some assumptions. For any compact interval J of R and for a nonnegative integer k, let $C^k(J)$ denote the space of all k-times continuously differentiable complex valued functions defined on J. For a function u, let $u^{(j)}$ denote the j^{th} derivative of u, if it exists. For a compact interval J of R and for a positive continuous (weight) function $r(x)$ defined on J, let $L(J,r)$ denote the Hilbert space of all

Lebesgue measurable complex valued functions u defined on J such that $r(x)|u(x)|$ is integrable over J. The inner product in $L(J,r)$ is given by $\langle u,v\rangle = \int u(x)\overline{v(x)}r(x)dx,\ u,v \in L(J,r)$, where $\overline{v(x)}$ denotes the complex conjugate of $v(x)$, and the norm is given by

$$\|u\| = \left(\int r(x)|u(x)|^2dx\right)^{1/2}, \quad u \in L^2(J,r).$$

Let $H^k(J,r)$ denote those functions in $AC^k(J)$ such that both u and $u^{(k)}$ are in $L^2(J,r)$. Let C^k denotes the k-dimensional complex space whose elements we take to be column vectors. For $k_1 \times k_2$ matrix A with complex entries, A^* denote the $k_2 \times k_1$ matrix which is the conjugate transpose of A. Let A^{-1} denote the inverse of a square matrix A, if it exists. If V_1 and V_2 are vector spaces (over the same field), then $V_1 \times V_2$ denotes the cartesian product of V_1 and V_2 taken in that order. For an operator T, $D(T), R(T), N(T), \eta(T)$ denote the domain, range, null space and the dimension of the null space of T, respectively. Let $X = L(J_1,r_1) \times L(J_2,r_2)$ be the cartesian product Hilbert space equipped with the inner product $\langle\cdot\rangle$ and the norm $\|\cdot\|$ given by

$$\langle\{u_1,u_2\},\{v_1,v_2\}\rangle = \langle u_1,v_1\rangle + \langle u_2,v_2\rangle, \{u_1,u_2\},\{v_1,v_2\} \in X,$$

and

$$\|\{u_1,u_2\}\| = (\|u_1\|^2 + \|u_2\|^2)^{1/2}, \quad \{u_1,u_2\} \in X.$$

Let $H = H^n(J_1,r_1) \times H^m(J_2,r_2)$ be the cartesian product Banach space.

ASSUMPTION 1. Let $J_1 = [a,b]$ and $J_2 = [b,c]$, $-\infty < a < b < c < +\infty$. Let $\tau_1 = \frac{1}{r_1}\sum_{k=0}^{n} P_kD^k$ and $\tau_2 = \frac{1}{r_2}\sum_{k=0}^{m} Q_kD^k$, be two formal differential expressions, where $P_k \in C^k(J_1)$, $k = 0,1,\ldots,n$, $P_n(x) \neq 0$ on J_1; $Q_k \in C^k(J_2)$, $k = 0,1,\ldots,m, Q_m(x) \neq 0$ on J_2; and $r_1(x) \in C^0(J_1)$ and $r_2(x) \in C^0(J_2)$ are positive real valued functions. We also assume $n \geq m$.

ASSUMPTION 2. Let A and B be $m \times n$ and $m \times m$ matrices with complex entries, respectively such that the range of A = range of B, and hence, rank of A = rank of $B = m$.

In Section 1, we shall collect together a few definitions and results, from our earlier papers, which we require here. In Section 2, we shall present a lemma regarding the form of solutions of a type of initial value problems (IVPs) associated with the pair (τ_1,τ_2), in terms of Green's matrices. In Section 3, we shall present an algorithm for the computation of Green's matrices for the BVPs associated with the pair (τ_1,τ_2). In Section 4, we shall construct the Green's matrices for problems encountered in the studies of acoustic wave guides in ocean and transverse vibrations in nonhomogeneous strings.

2. PRELIMINARIES

Let f be a complex valued function defined on J. Let $f_i = f/J_i$, $i = 1,2$. Let $J = J_1 \cup J_2$. Consider

$$(\tau_1,\tau_2)u = f \tag{2.1}$$

and the corresponding homogeneous equation

$$(\tau_1,\tau_2)u = 0. \tag{2.2}$$

DEFINITION 1. We call a complex valued function $u(x)$ defined on the interval J, a solution (nonexplicitly mixed) of (2.1) if

(i) the functions $u/J_1 = u_1 \in AC^n(J_1)$ and $u/J_2 = u_2 \in AC^m(J_2)$

(ii) u_1 and u_2 satisfy the equations $\tau_1 u_1 = f_1$, for $x \in J_1$ a.e., and $\tau_2 u_2 = f_2$, for $x \in J_2$ a.e., respectively.

DEFINITION 2. We call a complex valued function $u(x)$ defined on the interval J, a continuous solution of (2.1) if

(i) u is a solution of (2.1) in the sense of Definition 1, and

(ii) the functions u_1 and u_2 satisfy the continuity conditions at the interface point $x = b$, namely,

$$u_1^{(j)}(b-) = u_2^{(j)}(b+), \quad j = 0, 1, \ldots, m-1 .$$

DEFINITION 3. We call a complex valued function $u(x)$ defined on the interval J, a matching solution of (2.1) if

(i) u is a solution of (2.1) in the sense of Definition 1, and

(ii) the functions u_1 and u_2 satisfy certain matching conditions at the interface point $x = b$, namely, $A\vec{u}_1(b) = B\tilde{\vec{u}}_2(b)$, where

$$\vec{u}_1(b) = \text{column}(u_1(b), u_1^{(1)}(b), \ldots, u_1^{(n-1)}(b)),$$

and

$$\tilde{\vec{u}}_2(b) = \text{column}(u_2(b), u_2^{(1)}(b), \ldots, u_2^{(m-1)}(b)).$$

REMARK 1. All the above definitions can be carried over to equation (2.2) also.

Below, we recall a few definitions from [6], in the form, required here. Let $\tau = (\tau_1, \tau_2)$.

DEFINITION 5. The nonexplicitly mixed operator $N(\tau)$ is defined by

$$D(N(\tau)) = \{\{u_1, u_2\} \in H / B_i^N(\{u_1, u_2\}) = 0, \ i = 1, \ldots, n+m\},$$

$$N(\tau)\{u_1, u_2\} = \{\tau_1 u_1, \tau_2 u_2\},$$

where

$$B_i^N(\{u_1, u_2\}) = \sum_{j=0}^{n-1}(\alpha_{ij} u_1^{(j)}(a) + \beta_{ij} u_1^{(j)}(b)) + \sum_{j=0}^{m-1}(\gamma_{ij} u_2^{(j)}(b) + \delta_{ij} u_2^{(j)}(c)), \quad i = 1, \ldots, n+m$$

are the linearly independent nonexplicitly mixed boundary values.

DEFINITION 6. The continuously mixed operator $C(\tau)$ is defined by

$$D(C(\tau)) = \{\{u_1, u_2\} \in H / B_i^C(\{u_1, u_2\}) = 0, \ i = 1, \ldots, n, \ u_1^{(j)}(b) = u_2(j)(b), \ i = 1, \ldots, n\},$$

$c(\tau)\{u_1, u_2\} = \{\tau_1 u_1, \tau_2 u_2\})$ where

$$B_i^C(\{u_1, u_2\}) = \sum_{j=0}^{N-1}(\alpha_{ij} u_1^{(j)}(a) + \beta_{ij} u_1^{(j)}(b) + \delta_{ij} u_2^{(j)}(c)) \quad i = 1, \ldots, n,$$

are the linearly independent continuously mixed boundary values.

DEFINITION 7. The matchingly mixed operator $M(\tau)$ is defined by

$$D(M(\tau)) = \{\{u_1, u_2\} \in H / B_i^M(\{u_1, u_2\}) = 0, \ i = 1, \ldots, n+m, \ A\vec{u}_1(b) = B\tilde{\vec{u}}_2(b),\},$$

$$M(\tau)\{u_1, u_2\} = \{\tau_1 u_1, \tau_2 u_2\},$$

where

$$B_i^M(\{u_1, u_2\}) = \sum_{j=0}^{N-1}(\alpha_{ij} u_1^{(j)}(a) + \delta_{ij} u_2^{(j)}(c)) + \beta_i \cdot \vec{u}(b) \quad i = 1, \ldots, n,$$

are the linearly independent matchingly mixed boundary values.

REMARK 2. For the sake of brevity, we shall study only the operators $N(\tau)$ and $M(\tau)$ and the results for the operator $C(\tau)$ follow by taking $A = B = I$ (the $n \times n$ identity matrix) in the results for $M(\tau)$.

ASSUMPTION 3. For the matchingly mixed case we assume that $n = m$.

2. LEMMA REGARDING THE IVPs ASSOCIATED WITH (τ_1, τ_2)

We consider a particular type of initial value problem associated with (τ_1, τ_2) for nonexplicitly mixed and matchingly mixed operators and give a result about the form of the solution of the IVPs, in terms of Green's matrices.

(I) Nonexplicitly Mixed Initial Value Problems

Let $u_{11}, \ldots, u_{nl}$ and $u_{12}, \ldots, u_{m2}$ be functions in $H(J_1, r_1)$ and $H^m(J_2, r_2)$ which form bases for the solution spaces of $\tau_1 u_1 = 0$ and $\tau_2 u_2 = 0$, respectively. Then, the pairs $\{u_{11}, 0\}$, $\{u_{21}, 0\}, \ldots, \{u_{n1}, 0\}, \{0, u_{12}\}, \ldots, \{0, u_{m2}\}$ (all of which belong to H) form basis for the solution space of $N(\tau)\{u_1, u_2\} = 0$ (for the explicit form of the basis see [3]).

Define $N^0(\tau)$ to be the operator in H such that

$$N^0(\tau) = \{\{u_1, u_2\} \in H / u_1^{(j)}(a) = 0,\ j = 0, \ldots, n-1,\ u_2^{(j)}(b) = 0,\ j = 0, \ldots m-1\},$$

$$N^0(\tau)\{u_1, u_2\} = \{\tau_1 u_1, \tau_2 u_2\}.$$

REMARK 3. We note that the Wronskian of $u_{11}, \ldots, u_{n1}$, namely, $W(u_{11}, \ldots, u_{n1})(s) = 0$ for all $s \in J_1$, and the Wronskian of $u_{21}, \ldots, u_{m2}$, namely $W(u_{21}, \ldots, u_{m2})(s) = 0$ for all $s \in J_2$. And, we denote by $W_i(u_{11}, \ldots, u_{n1})(s)$ the determinant obtained by replacing the i^{th} column in the corresponding Wronskian by $(0, 0, \ldots, 1) \in C^n, i = 1, \ldots, n$. Similarly, we define $W_i(u_{12}, \ldots, u_{m2})(s)$.

(II) Matchingly mixed initial value problems

Let the set of pairs $\{u_{11}, u_{12}\}, \ldots, \{u_{n1}, u_{n2}\}$ be a basis for the solution space of $M_0(\tau)\{u_1, u_2\} = 0$, where

$$M_0(\tau) = \{\{u_1, u_2\} \in H / A\tilde{u}_1(b) = Bu_2(b)\},$$

$$M_0(\tau)\{u_1, u_2\} = \{\tau_1 u_1, \tau_2 u_2\}.$$

Also, define $M^0(\tau)$ to be the operator in H such that

$$M^0(\tau) = \{\{u_1, u_2\} \in H / u_1^{(j)}(a) = 0, j = 0, \ldots, n-1, A\tilde{u}_1(b) = Bu_2(b)\},$$

$$M^0(\tau)\{u_1, u_2\} = \{\tau_1 u_1, \tau_2 u_2\}.$$

The lemma below follows from the variation of parameters formula.

LEMMA. (I) The solution $\{u_1, u_2\}$ of $N^0(\tau)\{u_1, u_2\} = \{f_1, f_2\}$ is of the form $u(x) = \{u_1(x), u_2(x)\}$

$$= \begin{cases} \int_a^x G_{11}^{oN}(x,s) f_1(s) r_1(s)\, ds, & x \in J_1 \\ \int_b^x G_{22}^{oN}(x,s) f_2(s) r_2(s)\, ds, & x \in J_2 \end{cases}$$

where

$$G_{11}^{oN}(x,s) = \sum_{i=1}^{n} \frac{w_i(u_{11},\ldots\ldots,u_{nl})(s)}{P_n(s)W(u_{11},\ldots,u_{nl})(s)} u_{i1}(x), \quad a < s < x < b ,$$

$$G_{22}^{oN}(x,s) = \sum_{i=1}^{m} \frac{W_i(u_{12},\ldots\ldots,u_{m2})(s)}{Q_m(s)W(u_{12},\ldots,u_{m2})(s)} u_{i2}(x), \quad b < s < x < c ,$$

Also, we define

$$G^{oN} = \begin{bmatrix} G_{11}^{oN} & 0 \\ 0 & G_{22}^{oN} \end{bmatrix} \tag{2.3}$$

and call G^{oN} as the Green's matrix for $N^o(\tau)$.

II) The solution $\{u_1,u_2\}$ of $M^o(\tau)\{u_1,u_2\} = \{f_1,f_2\}$ is of the form $u(x) = \{u_1(x),u_2(x)\}$

$$= \begin{cases} \int_a^x G_{11}^{oM}(x,s)f_1(s)r_1(s)\,ds , & x \in J_1 \\ \int_a^b G_{12}^{oM}(x,s)f_1(s)r_1(s)\,ds + \int_b^x G_{22}^{oM}(x,s)f_2(s)r_2(s)\,ds , & x \in J_2 \end{cases}$$

where

$$G_{11}^{oM}(x,s) = \sum_{i=1}^{n} \frac{W_i(u_{11},\ldots\ldots,u_{n1})(s)}{P_n(s)W(u_{11},\ldots,u_{n1})(s)} u_{i1}(x), \quad a < s < x < b ,$$

$$G_{21}^{oM}(x,s) = \sum_{i=1}^{n} \frac{W_i(u_{11},\ldots\ldots,u_{n1})(s)}{P_n(s)W(u_{11},\ldots,u_{n1})(s)} u_{i2}(x), \quad a < s < b , \quad b < x < c ,$$

and

$$G_{22}^{oM}(x,s) = \sum_{i=1}^{m} \frac{W_i(u_{12},\ldots\ldots,u_{m2})(s)}{Q_m(s)W(u_{12},\ldots,u_{m2})(s)} u_{i2}(x), \quad b < s < x < c ,$$

Also, we define

$$G^{oM} = \begin{bmatrix} G_{11}^{oM} & 0 \\ G_{21}^{oM} & G_{22}^{oM} \end{bmatrix} \tag{2.4}$$

and call G^{oM} as the Green's matrix for $M^o(\tau)$.

3. COMPUTATIONAL ALGORITHM FOR THE GREEN'S MATRICES FOR OPERATORS ASSOCIATED WITH (τ_1,τ_2)

In this section, proceeding along the lines of [5], we present an algorithm for the computation of Green's matrices for operators associated with (τ_1,τ_2).

(I) Nonexplicitly mixed operator: Consider $\{f_1,f_2\} \in X$.

Let $u(x) = \{u_1(x),u_2(x)\} = (N^o(\tau))^{-1}\{f_1,f_2\}$. Then (see [4]), $u(x) = \{u_1(x),u_2(x)\}$

$$= \begin{cases} \int_a^b G_{11}^{N}(x,s)f_1(s)r_1(s)\,ds + \int_b^c G_{12}^{N}(x,s)f_1(s)r_1(s)\,ds , & x \in J_1 \\ \int_a^b G_{21}^{N}(x,s)f_1(s)r_1(s)\,ds + \int_b^c G_{22}^{N}(x,s)f_2(s)r_2(s)\,ds , & x \in J_2 . \end{cases} \tag{3.1}$$

we denote

$$G^N = \begin{bmatrix} G_{11}^N & G_{12}^N \\ G_{21}^N & G_{22}^N \end{bmatrix} \tag{3.2}$$

and we call G^N the Green's matrix for the operator $N(\tau)$. Let $v(x) = \{v_1(x), v_2(x)\} = (N^o(\tau))^{-1}\{f_1(x), f_2(x)\}$. By Theorem 4 [1], we have $\eta(N(\tau)) = n+m$. Since, $\{u_1 - v_1, u_2 - v_2\}$ belongs to the solution space of $\tau\{u_1, u_2\} = 0$, there exists scalars $c_1, \ldots, c_{n+m}$ such that $u(x) = \{u_1(x), u_2(x)\}$,

$$= \left\{ \sum_{i=1}^{n} c_i u_{i1}(x) + v_1(x), \sum_{i=1}^{m} c_{n+i} u_{i2}(x) + v_2(x) \right\} \tag{3.3}$$

Applying the boundary value on (3.3), we have $B_l^N(\{u_1, u_2\}) = 0 \quad l = 1, 2, \ldots, n+m$. That is,

$$B_l^N\left(\left\{ \sum_{i=1}^{n} c_i u_{i1}(x), \sum_{i=1}^{m} c_{n+i} u_{i2}(x) \right\} \right) = -B_l^N(\{v_1, v_2\}) \tag{3.4}$$

But,

$$B_l^N\left(\left\{ \sum_{i=1}^{n} c_i u_{i1}(x), \sum_{i=1}^{m} c_{n+i} u_{i2}(x) \right\} \right) = \sum_{i=1}^{n} c_i B_{li}^1 + \sum_{i=1}^{m} c_{n+i} B_{li}^2 ,$$

where

$$B_{li}^1 = \sum_{l=1}^{n-1} (\alpha_{jl} u_{i1}^{(j)}(a) + \beta_{jl} u_{i1}^{(j)}(b)) \quad i = 1, \ldots, n ,$$

and

$$B_{li}^2 = \sum_{l=1}^{m-1} (\gamma_{jl} u_{i1}^{(j)}(a) + \delta_{jl} u_{i1}^{(j)}(b) , \quad i = 1, \ldots, m , \quad l = 1, \ldots, n+m .$$

Relation (3.4) can now be written as,

$$\sum_{i=1}^{n} c_i B_{li}^1 + \sum_{i=1}^{m} c_{n+i} B_{li}^2 = -B_l^N(\{v_1, v_2\}) , \quad l = 1, \ldots, n+m . \tag{3.5}$$

It can be verified that the coefficient matrix of the $(n+m) \times (n+m)$ linear system (3.5) in $(n+m)$ unknowns, is nonsingular. Now, by the choice of $\{v_1, v_2\}$, we have

$$B_l^N(\{v_1, v_2\}) = \sum_{k=0}^{n-1} \beta_{kl} v_{i1}^{(k)}(b) + \sum_{k=0}^{m-1} \delta_{jl} v_{i1}^{(k)}(c)$$

$$= \int_a^b \mathcal{H}_{l1}(s) f_1(s) r_1(s) \, ds + \int_b^c \mathcal{H}_{l2}(s) f_2(s) r_2(s) \, ds$$

where

$$\mathcal{H}_{l1}(s) = \sum_{j=1}^{n} \sum_{k=0}^{n-1} \frac{W_j(u_{11}, \ldots, u_{n1})(s)}{P_n(s) W(u_{11}, \ldots, u_{n1})(s)} (\beta_{lk} u_{lk}^{(k)}(b)) ,$$

and

$$\mathcal{H}_{l2}(s) = \sum_{j=1}^{m} \sum_{k=0}^{m-1} \frac{W_j(u_{12}, \ldots, u_{m2})(s)}{Q_m(s) W(u_{12}, \ldots, u_{m2})(s)} (\delta_{lk} u_{l2}^{(k)}(c))$$

$l = 1, \ldots, n+m$. Clearly, $\mathcal{H}_{l1} \in H^n(J_1, r_1)$ and $\mathcal{H}_{l2} \in H^m(J_2, r_2)$. Rewriting (3.5), we have, for $l = 1, \ldots, n+m$,

$$\sum_{i=1}^{n} c_i B_{li}^1 + \sum_{i=1}^{m} c_{n+i} B_{li}^2 = -\int_a^b \mathcal{H}_{l1}(s) f_1(s) r_1(s) \, ds - \int_b^c \mathcal{H}_{l2}(s) f_2(s) r_2(s) \, ds . \tag{3.6}$$

Let $B^1 = (B^1_{li}), i = 1,\dots,n$ and $B^2 = (B^2_{lj}), j = 0,\dots,m$ $l = 1,\dots,n+m$. Let $B = [B^1, B^2]$. It can be shown that B is a nonsingular matrix. That is, $det\, B = 0$. Consider the $(n+m)\times(n+m)$ linear system, for $l = 1,\dots,n+m$,

$$\sum_{i=1}^{n} B^1_{li}\{z_{i1}, z_{i2}\} + \sum_{i=1}^{m} B^2_{li}\{z_{(n+i)1}, z_{(n+i)2}\} = -\{\mathcal{H}_{l1}, \mathcal{H}_{l2}\}\,, \tag{3.7}$$

We have by Cramer's rule, $\{z_{i1}(s), z_{i2}(s)\} = \frac{-1}{det\, B}\{B_{i1}, B_{j2}\}, i = 1,\dots,n$, and $j = 1,2,\dots,m$, where B_{il} and B_{j2} are determinants obtained by replacing the i^{th} and j^{th} columns in B^1_{li} and B^2_{lj}, by the column vectors $(\mathcal{H}_{11},\dots,\mathcal{H}_{(n+m)1})$ and $(\mathcal{H}_{12},\dots,\mathcal{H}_{(n+m)2})$, respectively. That is, each of $z_{i1}(s)$ and $z_{i2}(s)$ are linear combinations of $\mathcal{H}_{i1}(s)$ and $\mathcal{H}_{i2}(s)$, respectively. Hence, $\{z_{il}(s), z_{i2}(s)\} \in H$. Then, we have by taking the inner-product of both sides of (3.7) with $\{f_1, f_2\}$,

$$\begin{aligned}
&\sum_{i=1}^{n} B^1_{li}\langle\{z_{il}, z_{i2}\}, \{f_1, f_2\}\rangle + \sum_{i=1}^{m} B^2_{li}\langle\{z_{(n+1)i}, z_{(n+i)2}\}, \{f_1, f_2\}\rangle \\
&= -\langle\{\mathcal{H}_{il}, \mathcal{H}_{i2}\}, \{f_1, f_2\}\rangle \\
&= \sum_{i=1}^{n} c_i B^1_{li} + \sum_{i=1}^{m} c_{n+i} B^2_{li}\,,
\end{aligned}$$

(by (3.6)), which implies that,

$$c_i = \langle\{z_{il}(s), z_{i2}(s)\}, \{f_1(s), f_2(s)\}\rangle\,, \quad i = 1,\dots,n+m\,. \tag{3.8}$$

Combining (3.3) and (3.8), and comparing with (5), we get,

$$G^{oN}_{11}(x,s) = \begin{cases} \sum_{i=1}^{n} u_{il}(x) z_{il}(s) + G^{oN}_{11}(x,s)\,, & a < x < s < b \\ \sum_{i=1}^{n} u_{il}(x) z_{il}(s) & , \ a < x < s < b \end{cases}$$

$$G^{oN}_{12}(x,s) = \sum_{i=1}^{n} u_{il}(x) z_{il}(s)\,, \quad a < x < b\,, \quad b < s < c$$

$$G^{oN}_{21}(x,s) = \sum_{i=1}^{m} u_{i2}(x) z_{i2}(s)\,, \quad b < x < c\,, \quad a < s < b$$

$$G^{oN}_{22}(x,s) = \begin{cases} \sum_{i=1}^{m} u_{i2}(x) z_{i2}(s) + G^{oN}_{22}(x,s)\,, & b < s < x < c \\ \sum_{i=1}^{m} u_{i2}(x) z_{i2}(s) & , \ b < x < s < c \end{cases}$$

This completes the algorithm for the computation of Green's matrix G^N for the nonexplicitly mixed operator $N\tau$).

REMARK 4. The algorithm for the computation of the Green's matrix G^M for the operator $M(\tau)$, runs along the similar lines, with $n = m$.

4. PHYSICAL EXAMPLES

In this section, we shall use the computational algorithms developed in Section 3, to compute the Green's matrices for a matchingly mixed operator and a continuously mixed operator, encountered in the studies of acoustic waveguides in oceans and transverse vibrations in nonhomogeneous strings, respectively.

(I) Acoustic waveguides in oceans [6]:

Consider the ocean to be consisting of two homogeneous layers, with a rigid bottom and a pressure release surface. Then, the propagation of acoustic waveguides in such an ocean is governed by the following equations.

$$\tau_1 u_1 = u_1^{(2)} + K_1^2 u_1 = \lambda u_1 , \quad 0 < x < d_1$$

$$\tau_2 u_2 = u_2^{(2)} + K_2^2 u_2 = \lambda u_2 , \quad d_1 < x < d_2$$

together with the mixed boundary conditions given by,

$$u_1(0) = u_2^{(1)}(d_2) = 0 , \quad u_1(d_1) = u_2(d_2) , \quad \frac{1}{\rho_1} u_1(d_1) = \frac{1}{\rho_2} u_2(d_1) ,$$

where ρ_1 and ρ_2 are constant densities of the two layers, K_1, K_2 are constants which depend upon the frequency constant ω and the constant sound velocities c_1, c_2 of the two layers, respectively, λ is an unknown constant, $[0, d_1]$ and $[d_1, d_2]$ denote the depth of the two layers and u_1 and u_2 stand for the depth eigenfunctions. Let $J_1 = [0, d_1]$ and $J_2 = [d_1, d_2]$. The matching conditions at the interface $x = d_1$ can be written in the matrix form $A_1 \tilde{u}_1(d_1) = B_2 \tilde{u}_2(d_1)$, where $\tilde{u}_i(d_1) = \text{column}(u_i(d_1), u_i^{(1)}(d_1))$, $A_i = \begin{pmatrix} 1 & 0 \\ 0 & 1/\rho_i \end{pmatrix}$ for $i = 1, 2$. Also, we have $n = m = d = 2$. Define

$$M(\tau) = \{\{u_1, u_2\} \in H^2(J_1, 1/\rho_1) \times H^2(J_2, 1/\rho_2) / A_1 \tilde{u}_1(d_1) = A_1 \tilde{u}_2(d_1), u_1(0) = u_2^{(1)}(d_2) = 0\} ,$$

$$M(\tau) u = \{\tau_1 u_1, \tau_2 u_2\} .$$

After simple calculations along lines of the algorithm, we get the form of the Green's matrix G^M to be of the form,

$$G_{11}^M = \begin{cases} \dfrac{\sin K_1 x}{K_1 M} (\rho_2 K_1 \cos K_2(d_2 - d_1) \cos K_1(d_1 - s) - \rho_1 K_2 \sin K_1(d_1 - s) \sin K_2(d_2 - d_1)) , & 0 < x < s < d_1 , \\ \dfrac{\sin K_1 s}{K_1 M} (\rho_2 K_1 \cos K_2(d_2 - d_1) \cos K_1(d_1 - x) - \rho_1 K_2 \sin K_1(d_1 - x) \sin K_2(d_2 - d_1)) , & 0 < s < x < d_1 , \end{cases}$$

$$G_{12}^M = \frac{\rho_1}{M} \sin K_1 x \cos K_2(d_2 - s) , \quad 0 < x < d_1 , \quad d_1 < s < d_2$$

$$G_{12}^M = \frac{\rho_2}{M} \sin K_1 s \cos K_2(d_2 - x) , \quad 0 < s < d_1 , \quad d_1 < x < d_2$$

$$G_{11}^M = \begin{cases} \dfrac{\cos K_2(d_2 - s)}{K_1 M} (\rho_2 K_1 \cos K_2(d_2 - d_1) \cos K_1(d_1 - s) - \rho_2 K_1 \cos K_1 d_1 \sin K_2(x - d_1)) , & d < x < s < d , \\ \dfrac{\cos K_2(d_2 - x)}{K_2 M} (\rho_2 K_2 \sin K_1 d_1 \cos K_2(d_1 - s) + \rho_2 K_1 \cos K_1 d \sin K_2(s - d_1)) , & d_1 < s < x < d_2 , \end{cases}$$

We also note that

$$A_1 \tilde{G}_{11}^M(d_1, s) = A_2 \tilde{G}_{21}^M(d_1, s)$$

and

$$A_1 \tilde{G}_{12}^M(d_1, s) = A_2 \tilde{G}_{22}^M(d_1, s)$$

REMARK 5. In the above, we have the compact and general form of the Green's matrix of the problem compared to the one given in [6].

(II) Transverse vibrations in nonhomogeneous strings [7]:

Consider the string consisting of two portions of lengths d_1 and $d_2 - d_1$, and different uniform densities ρ_1, ρ_2 respectively, having tension T and stretched between the points $x = 0$ and $x = d_2$. The modes of transverse vibrations of the above string are governed by,

$$\tau_1 u_1 = c_1^2(-u_1^{(2)}) = \lambda u_1, \quad 0 < x < d_1,$$

and

$$\tau_2 u_2 = c_2^2(-u_2^{(2)}) = \lambda u_2, \quad d_1 < x < d_2,$$

together with the mixed boundary conditions given by,

$$u_1(0) = u_2(d_2) = 0, \quad u_1(d_1) = u_2(d_1), \quad u_1^{(1)}(d_1) = u_2^{(1)}(d_1),$$

where $c_i^2 = T/\rho_i$, $i = 1,2$. Here, the conditions at the interface point are the continuity conditions.

Proceeding along the lines of the algorithm, we get, after routine calculations, the Green's matrix G to be of the form,

$$G^C = \begin{bmatrix} \dfrac{(s-d_2)x}{c_1^2 d_2}, \quad 0<x<s<d_1 & \dfrac{(s-d_2)x}{c_2^2 d_2}, \quad 0<x<d_1, \quad d_1<s<d_2 \\ \dfrac{(x-d_2)s}{c_2^2 d_2}, \quad 0<s<x<d_1 & \dfrac{(x-d_2)s}{c_2^2 d_2}, \quad 0<x<s<d_1 \\ \dfrac{(x-d_2)s}{c_1^2 d_2}, \quad 0<s<d_1, \quad d_1<x<d_2 & \dfrac{(s-d_2)x}{c_2^2 d_2}, \quad 0<s<x<d_1 \end{bmatrix}$$

We note that $\tilde{G}_{11}^C(d_1,s) = \tilde{G}_{21}^C(d_1,s)$ and similar relations are true of the components G_{12}^C and G_{22}^C.

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