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ABSTRACT. The radius of univalence is found for the convolution $f*g$ of functions $f \in S$ (normalized univalent functions) and $g \in C$ (close-to-convex functions). A lower bound for the radius of univalence is also determined when f and g range over all of S. Finally, a characterization of C provides an inclusion relationship.

KEY WORDS AND PHRASES. Univalent, convolution.
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1. INTRODUCTION.

Denote by S the family consisting of functions $f(z) = z + \cdots$ that are analytic and univalent in $\Delta = \{z: \mid z \mid < 1\}$ and by K, S^*, and C the subfamilies of functions that are, respectively, convex, starlike, and close-to-convex in Δ. It is well known that $K \subset S^* \subset C \subset S$. The convolution of two power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n$$

is defined as the power series

$$f(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

The Koebe function $k(z) = z/(1-z)^2$ often plays an extremal role in the family S. This enables us to show it to be extreme in many convolution problems. For example, the modulus of the nth coefficient for $f*g$, f and g in S, is n^2 and is attained when $f = g = k$. Similarly, $\mid f*g \mid$ takes its maximum and minimum on the circle $\mid z \mid = r$ when $f = g = k$.

A question was raised in [4] as to whether

$$\min_{\mid z \mid = r} Re \frac{(f*g)(z)}{z} = \min_{\mid z \mid = r} Re \frac{(f*k)(z)}{z} = \min_{\mid z \mid = r} Re f'$$

when f and g are taken over all of S. The classical rotation theorem for $f \in S$ leads to the sharp result that $Re f'(z) \geq 0$ when $\mid z \mid \leq sin(\pi/8)$. This was generalized in [4] to $Re \frac{(f*g)(z)}{z} \geq 0$ for $\mid z \mid \leq sin(\pi/8)$ when $f \in S$ and $g \in S^*$, but could not be extended to $g \in S$ or even to $g \in C$. In particular, functions $f, g \in C$ were found for which $Re \frac{(f*g)(z)}{z} < 0$ at some point z_0, $\mid z_0 \mid < sin(\pi/8)$.

In this note, we investigate the radius of univalence for $f*g$, f and g in S For $f \in S$ and $g = k$, the Koebe function, $f*g$ is univalent in the disk $|z| < 2 - \sqrt{3}$ We prove that $g = k$ can be replaced by any $g \in C$, but we cannot settle if this extends to arbitrary $g \in S$. We do show, however, that $f*g$ is univalent for at least $|z| < .8(2-\sqrt{3})$

2. MAIN RESULTS.

THEOREM 1. If $f \in S$ and $g \in C$, then $f*g$ is univalent in $|z| < 2 - \sqrt{3}$. The result is sharp

PROOF. It is well known that f is convex in $|z| < r$ if and only if zf' is starlike in $|z| < r$ and that the radius of convexity of S is $2-\sqrt{3}$. Thus, $f*k = zf'$ has radius of starlikeness (and hence radius of univalence) at least $2-\sqrt{3}$, the radius of convexity for $f \in S$ Since

$$(k*k)' = (zk')' = \frac{1+4z+z^2}{(1-z)^4} = 0 \text{ at } z = -(2-\sqrt{3}),$$

the radius of univalence of $f*g$ for $f, g \in S$ can be no greater than $r = 2-\sqrt{3}$.

When $f \in S$, we have $f(az)/a \in K$ for $a = 2-\sqrt{3}$. Hence, by a theorem of Ruscheweyh and Sheil-Small [3], if $f \in S$ and $g \in C$ then

$$\frac{f(az)}{a} * g(z) \in C \subset S.$$

Thus, $f*g$ is univalent for $|z| < 2-\sqrt{3}$, and the proof is complete.

In our next theorem, we replace C with S in the hypothesis and this leads to a weaker conclusion.

THEOREM 2. Denote by r_0 the largest value for which $f*g$ is univalent in $|z| < r_0$ for all $f, g \in S$. Then $.8(2-\sqrt{3}) < r_0 \le 2-\sqrt{3}$.

PROOF. The upper bound was found in Theorem 1. Krzyz [1] determined the radius of close-to-convexity for S to be $t_0 = 0.80+$. Since $f(az)/a \in K, a = 2-\sqrt{3}$, and $g(t_0 z)/t_0 \in C$, we have from the Ruscheweyh and Sheil-Small theorem [3] that $\frac{f(az)}{a} * \frac{g(t_0 z)}{t_0} \in C$, which shows that $f*g$ is univalent for $|z| < t_0(2-\sqrt{3})$. This furnishes us with the lower bound, and the proof is complete.

Though we are unable to prove that $r_0 = 2-\sqrt{3}$ in Theorem 2, the lower bound on r_0 most certainly can be improved. Ruscheweyh defined the family M consisting of normalized functions f by

$$M = \{f: f*g \neq 0; g \in S^*, 0 < |z| < 1\}.$$

He proved the proper inclusions $C \subset M \subset S$ and that $f*g \in M$ for $f \in K$ and $g \in M$ [2]. Hence, if t_1 is the largest value for which $g(t_1 z)/t_1 \in M$ when $g = S$, methods identical to those of Theorem 2 show that $f*g$ is univalent in $|z| < t_1(2-\sqrt{3})$ for $f, g \in S$. Unfortunately the value of t_1, the radius of "M-ness" for S, is unknown.

3. A CHARACTERIZATION OF C.

The inclusion $C \subset M$ is not obvious and was proved by Ruscheweyh using his duality principle [2]. Our final result is a characterization of C that leads to a more elementary proof that $C \subset M$. We make use of a result found in [3].

LEMMA 3. If $\phi \in K$, $\Psi \in S^*$, and F is analytic with $Re F > 0$ for $z \in \Delta$, then

$$Re\, \frac{\phi * F\Psi}{\phi * \Psi} > 0.$$

THEOREM 3. A function $f \in C$ if and only if to each $g \in S^*$ we may associate an $h \in S^*$ for which $Re\ \frac{f*g}{h} > 0,\ z \in \Delta$.

PROOF. To show that the condition is sufficient for f to be in C, we choose $g(z) = z/(1-z)^2 \in S^*$. Then $Re\ \frac{f*g}{h} = Re\ \frac{zf'}{h} > 0$, which means that $f \in C$.

On the other hand, if $f \in C$ we can find a $\Psi = S^*$ for which $Re zf'/\Psi > 0$ Set $F(z) = zf'(z)/\Psi(z)$. Then for $g \in S^*$ there corresponds $\phi \in K$ such that $z\phi' = g$. Note that $f*g = zf'*\phi = \phi*F\Psi$ and that $h = \phi*\Psi \in S^*$. By Lemma A,

$$Re\ \frac{\phi*F\Psi}{\phi*\Psi} = Re\ \frac{f*g}{h} > 0,$$

and the proof is complete

COROLLARY. $C \subset M$

PROOF. Since $Re\ \frac{f*g}{h} > 0 \Rightarrow f*g \neq 0$, the result follows from Theorem 3.

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