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ELLIPTIC RIESZ OPERATORS ON THE WEIGHTED SPECIAL ATOM SPACES

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ABSTRACT. In this paper we study the boundedness and convergence of $\sigma_r^s(f)$ and $\tilde{\sigma}_r^s(f)$, the elliptic Riesz operators and the conjugate elliptic Riesz operators of order $s > 0$, on the weighted special atom space $B(\omega)$.

KEY WORDS AND PHRASES. Elliptic Riesz operators, weighted special atom space, Lorentz spaces.
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1. INTRODUCTION.

Let R^n be n-dimensional Euclidean space and Z^n be the unit lattice in R^n. The n-Torus T^n is the coset space $R^n/(2\pi Z^n)$, $Q^n = \{x = (x_1, .., x_n) : 0 < x_k \leq 2\pi, 1 \leq k \leq n\}$. Let $A(D)$ be a self-adjoint elliptic differential operator with real coefficients defined on $C_0^\infty(R^n)$, $A(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha$, where $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}$, $\alpha = (\alpha_1, ..., \alpha_n)$ is multi index and $|\alpha| = \alpha_1 + ... + \alpha_n$. We always assume that the set $\{x \in R^n : A(x) < 1\}$ is convex and its boundary has non-vanishing Gaussian curvature everywhere.

The elliptic Riesz operators and the conjugate elliptic Riesz operators of order $s > 0$ are defined respectively by

$$\sigma_r^s(f, x) = \sum_{m \in Z^n} (1 - A(m/r))_+^s \hat{f}(m) e^{imx} \tag{1.1}$$

$$\tilde{\sigma}_r^s(f, x) = \sum_{m \in Z^n} (1 - A(m/r))_+^s \hat{f}(m) \hat{K}(m) e^{imx} \tag{1.2}$$

where

$$\hat{f}(m) = (2\pi)^{-n} \int_{Q^n} f(x) e^{-imx} dx$$

are the multiple Fourier coefficients of f, $K(x) = P(x)/|x|^{n+m} (x \neq 0)$ is a kernel with a homogeneous and harmonic polynomial $P(x)$ of order m, and $\tilde{f}$ is the conjugate function of f with respect to the kernel $K(x)$. $\beta_+ = \max\{0, \beta\}$. If $A(\xi) = |\xi|^2$, $\sigma_r^s(f)$, $\tilde{\sigma}_r^s(f)$ is just the usual Bochner-Riesz mean.

The maximal elliptic Riesz operators defined by

$$\sigma^s(f, x) = \sup_{r > 0} |\sigma_r^s(f, x)|, \tilde{\sigma}^s(f, x) = \sup_{r > 0} |\tilde{\sigma}_r^s(f, x)| .$$

In this paper, using the weighted special atom space $B(\omega)$, we will study the boundedness and convergence of $\sigma_r^s(f)$ and $\tilde{\sigma}_r^s(f)$ for all $s > 0$ and $n = 1$.

We rewrite $B(\omega)$ which was introduced in [4]:

$$B(\omega) = \left\{ f : T \to R', f(t) = \sum_{k=0}^{\infty} C_k b_k(t), \sum_{k=0}^{\infty} |C_k| < \infty \right\},$$

each b_k is a weighted special atom, that is, a real valued function b, defined on $T = [0, 2\pi]$, which is either $b(t) = 1/(2\pi)$ or $b(t) = \omega(|Q|)^{-1/q}$. $[\chi_R(t) - \chi_L(t)]$, $1 \le q < \infty$, where Q is an interval in T, L is the left half of Q and R is the right half, $|Q|$ denotes the length of Q, χ_Q the characteristic function of Q and ω is a non-negative real valued function which is increasing, and $\omega(0) = 0$. $B(\omega)$ is endowed with the norm $\|f\|_{B(\omega)} = \inf\left\{ \sum_{k=0}^{\infty} |C_k| \right\}$, where the infimum is taken over all possible representations of f. $B(\omega)$ is a Banach space.

A function $\omega : [0, \infty) \to [0, \infty)$ is said to be in the class $b_\lambda (0 < \lambda < \infty)$, if it satisfies

(1) $\omega(0) = 0$,

(2) ω is non-decreasing,

$\omega(t)/t$ is decreasing,

(4) $\int_0^h \omega(t)/t dt \le C\omega(h)$, C an absolute constant,

(5) $\int_h^{2\pi} \omega(t)/t^{\lambda+1} dt \le C\omega(h)/h^\lambda$ with C independent of h and ω.

Example of functions in the class b_λ are $\omega(t) = t^\alpha$ $(0 < \alpha < 1)$ and $\omega(t) = t^\alpha(\log(e/t))^\beta$, $(0 < \alpha < \lambda, \beta \ge 0)$.

We also define the space $L(\phi)$ be $L(\phi) = \{f : T \to R', \|f\|_\phi < \infty\}$, where $\|f\|_\phi = (\int_T (f^*(t))^q \phi(t) dt)^{1/q}$, $1 \le q < \infty$ and f^* is the decreasing rearrangement of f, defined by $f^*(t) = \inf\{y : |\{x : |f(x)| > y\}| \le t\}$, the outside bars means the Lebesgue measure of the set $\{x : |f(x)| > y\}$, ϕ is a non-negative decreasing function. $\|\bullet\|_\phi$ is a norm if and only if ϕ is a non-negative decreasing function. $L(\phi)$ is a Banach space. If $\omega(t) = (q/p)t^{q/p}$, $1 \le q \le p < \infty$, $\phi(t) = \omega(t)/t$, then the space $L(\phi)$ is the Lorentz space $L(p, q)$ in [6,7].

The main result of this paper is stated as follows:

THEOREM 1. Suppose $\omega \in b_\lambda$, $1 \le \lambda < \infty$, $\phi(t) = \omega(t)/t$, then $\sigma^s(f)$ is of type $(B(\omega), L(\phi))$ for all $s > 0$, that is,

$$\|\sigma^s(f)\|_\phi \le C\|f\|_{B(\omega)}, \quad f \in B(\omega).$$

COROLLARY 1. Suppose $\omega \in b_\lambda$, $1 \le \lambda < \infty$, and $f \in B(\omega)$, then $\sigma_r^s(f, x)$ converges to $f(x)$ almost everywhere for all $s > 0$.

THEOREM 2. Suppose $\omega \in b_\lambda$, $1 \le \lambda < \infty$, $\phi(t) = \omega(t)/t$, then $\tilde{\sigma}^s(f)$ is of type $(B(\omega), L(\phi))$ for all $s > 0$, that is,

$$\|\tilde{\sigma}^s(f)\|_\phi \le C\|\tilde{f}\|_{B(\omega)} \le C\|f\|_{B(\omega)}, \quad f \in B(\omega).$$

COROLLARY 2. Suppose $\omega \in b_\lambda$, $1 \le \lambda < \infty$, and $f \in B(\omega)$, then $\tilde{\sigma}_r^s(f, x)$ converges to $\tilde{f}(x)$ almost everywhere for all $s > 0$.

REMARK 1. When $n = 1$, $A(\xi) = |\xi|^2$, $\sigma_r^s(f, x)$ become

$$\sigma_r^s(f, x) = \sum_{|k|<r} (1 - (|k|/r)^2)^s \hat{f}(k) e^{ikx}. \tag{1.3}$$

As $s \to 0$, (1.3) become the partial sums of Fourier series of f, when $s = 1/2$, (1.3) are essentially equivalent to the classical Cesàro means. Consequently, the main result in [5,6] become a special case of our results.

REMARK 2. For the maximal (C, α) operators T are defined by

$$T(f, x) = \sup_n |\sigma_n^\alpha(f, x)| \tag{1.4}$$

where

$$\sigma_n^\alpha(f,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)K_n^\alpha(x-t)dt,$$

since (C,α) kernels

$$K_n^\alpha(t) = \sum_{k=0}^{n} A_{n-k}^{\alpha-1}D_k(t)/A_n^\alpha \tag{1.5}$$

satisfies

$$|K_n^\alpha(t)| \le \begin{cases} \dfrac{An}{(1+nt)(1+(nt)^\alpha)} \le \dfrac{An}{1+nt}, & 0<\alpha<1,\ 0\le t\le\pi, \\ C/|t|, & \alpha=1,\ 0<|t|\le\pi, \end{cases}$$

thus using the same methods for $\phi(t) = \omega(t)/t$ we can prove

$$\|Tf\|_\phi \le C\|f\|_{B(\omega)}, \qquad f\in B(\omega),\ 0<\alpha\le 1.$$

2. PROOFS OF THEOREMS

PROOF OF THEOREM 1. Let $f^\alpha(x) = f(x-a)$, then the operator $T_a f = f^a$ is of type $(B(\omega), B(\omega))$. Consequently, we just need to prove the result for $f_h(t) = [\omega(2h)]^{-1/q}[\chi_{[-h,0)}(t) - \chi_{[0,h]}(t)], h>0$ which will follow from the estimate for $g(t) = \chi_{[0,h]}(t)$. Let $H(x) = (2\pi)^{-1}\int_{R^1}(1-A(y))_+^s e^{ixy}dy,\ s>0,\ H_{1/r}(x) = rH(rx)$, then $\sigma_r^s(f,x) = (f*K_{1/r})(x)$, where

$$K_{1/r}(x) = \sum_{k=-\infty}^{\infty} H_{1/r}(x+2k\pi).$$

We may assume $r>1$. By the inequality (see [2]):

$$|H(x)| \le C(1+|x|)^{-s-1},$$

we get

$$|K_{1/r}(x)| \le Cr\sum_{k=-\infty}^{\infty}(1+r|2k\pi+x|)^{-(s+1)} \le Cr(1+r|x|)^{-(s+1)}.$$

Thus

$$\begin{aligned}|\sigma_r^s(g,x)| &= \left|\int_{-\pi}^{\pi} g(y)K_{1/r}(x-y)dy\right| = \left|\int_0^h K_{1/r}(x-y)dy\right| \le \int_{x-h}^{x}|K_{1/r}(t)|dt \\ &\le C\int_{x-h}^{x} r(1+rt)^{-1}dt \le Ch(x-h)^{-1} < 2Ch/x,\end{aligned}$$

for $x>2h$, and $|\sigma_r^s(g,x)| \le -2Ch/x$ for $x<-2h$. On the other hand, we have

$$\begin{aligned}|\sigma_r^s(g,x)| &\le \int_0^h |K_{1/r}(x-y)|dy \le \int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty}|H_{1/r}(x+2k\pi-y)|\right)dy \\ &= \int_{-\infty}^{\infty}|H_{1/r}(y)|dy \le C\int_{-\infty}^{\infty}(1+|t|)^{-(s+1)}dt < \infty.\end{aligned}$$

Consequently, we have

$$|\sigma^s(g,x)| \le \begin{cases} A, & \text{for all } x, \\ 2Ch/|x|, & \text{for } |x|>2h. \end{cases} \tag{2.1}$$

Let $\phi(t) = \omega(t)/t$. By (2.1) and the conditions on ω, we get

$$\|\sigma^s(g)\|_\phi^q = \int_0^{2\pi} ((\sigma^s(g))^*(x))^q \omega(x)/x dx \le A^q \int_0^{2h} \omega(x)/x dx$$
$$+ (2Ch)^q \int_{2h}^{2\pi} \omega(x)/x^{(q+1)} dx \le CA^q \omega(2h) + (2Ch)^q (\omega(2h)/(2h)^q) = C\omega(2h).$$

The constant C may not be the same at every occurrence in this paper. Thus $\|\sigma^s(f_h)\|_\phi \le 2\omega(2h)^{-(1/q)} \bullet \|\sigma^s(g)\|_\phi \le C$ and so if $f \in B(\omega)$, then $f(t) = \sum_{k=0}^{\infty} C_k b_k(t)$, where

$$b_k(t) = \omega(|Q_k|)^{-1/q}[\chi_{R_k}(t) - \chi_{L_k}(t)]$$

and $\sum_{k=0}^{\infty} |C_k| < \infty$, we have $\|\sigma^s(f)\|_\phi \le C \sum_{k=0}^{\infty} |C_k|$, which implies $\|\sigma^s(f)\|_\phi \le C\|f\|_{B(\omega)}$. The proof is complete.

PROOF OF COROLLARY 1. Let

$$\omega(f,x) = \limsup_{r \to \infty} |\sigma^s_{r_1}(f,x) - \sigma^s_{r_2}(f,x)|, r_1, r_2 > r, \tag{2.2}$$

then $\omega(f,x) \le 2\sigma^s(f,x)$ and so

$$\|\omega(f)\|_\phi^q = \int_0^{2\pi} ((\omega(f))^*(x))^q \omega(x)/x dx \le 2 \int_0^{2\pi} ((\sigma^s(f))^*(x))^q \omega(x)/x dx$$
$$= 2\|\sigma^s(f)\|_\phi^q. \tag{2.3}$$

Since $f \in B(\omega)$, then $f(x) = \sum_{k=0}^{\infty} C_k b_k(x)$, where $\sum_{k=0}^{\infty} |C_k| < \infty$ and the b_k are weighted special atoms. By Theorem 1 and (2.3), $\sigma^s(f) \in L(\phi)$ which implies $\omega(f) \in L(\phi)$ for $\phi(t) = \omega(t)/t$. On the other hand, we see that $\omega(f) = \omega(f - f_m)$ where $f_m(x) = \sum_{k=0}^{m} C_k b_k(x)$ and $\|f_m - f\|_{B(\omega)} \to 0$ as $m \to \infty$. Then

$$\omega(f,x) = \omega(f - f_m, x) \le 2\sigma^s(f - f_m, x).$$

By Theorem 1,

$$\|\omega(f)\|_\phi \le 2\|\sigma^s(f - f_m)\|_\phi \le 2C\|f - f_m\|_{B(\omega)}.$$

So letting $m \to \infty$, we get $\|\omega(f)\|_\phi = 0$. Thus $\omega(f,x) = 0$ almost everywhere, which implies $\sigma^s_r(f,x)$ converges to $f(x)$ almost everywhere. The proof is complete.

Let $f \in B(\omega)$, then $f(x) = \sum_{k=0}^{\infty} C_k b_k(x)$, where $\sum_{k=0}^{\infty} |C_k| < \infty$ and the b_k are weighted special atoms. Thus $\tilde{f}(x) = \sum_{k=0}^{\infty} C_k \tilde{b}_k(x)$ and so $\|\tilde{f}\|_{B(\omega)} \le \|f\|_{B(\omega)}$. Now using $\tilde{\sigma}^s_r(f,x) = \sigma^s_r(\tilde{f},x)$ and Theorem 1, we can similarly show that Theorem 2 and Corollary 2. The details will be omitted.

REMARK 3. Theorem 1 and 2 are also true if we replace the above $\omega(|Q|)$ by a weight $\omega(Q) = \int_Q \omega(x)dx$, where ω in A_∞, the proofs are the same.

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