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TWO CRITERIA FOR UNIVALENCY

N. SAMARIS

Department of Mathematics
University of Patras
Patras 26110, Greece

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ABSTRACT. In the present paper we give two criteria for the functions $f(z) = z + a_2z^2 + ...$ to be univalent in $|z| < 1$

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Let A denote the class of functions which are analytic in the unit disk $U = \{z : |z| < 1\}$ and $f'(0) - 1 = 0$. By B we denote the class of functions $f \in A$ which are univalent, convex and bounded in U. In the present paper we prove the following theorems.

THEOREM 1. Let $f \in A$, satisfy the condition

$$|\frac{q_0^2}{q'}[\frac{z^2 f'}{f^2} - \frac{z^2 q'}{q^2}]| \leq 1 \text{ in } U. \tag{1}$$

for some $q \in B$, where $q_0 = sup\{|q(z)|; z \in U\}$. Then f is univalent in U.

THEOREM 2. Let $f \in A, q \in B$ satisfy the condition

$$|\frac{1}{q'}[\frac{z}{f(z)} - \frac{z}{q(z)}]| \leq \lambda \text{ in } U, \tag{2}$$

for some $q \in B$, where

$$\lambda = inf\{|\frac{q'(z)}{q_0^2 q(z)}|; z \in U\}.$$

Then f is univalent in U.

If we put $q(z) = z$ in the Th.1 we get the Ozaki and Nunokawa theorem [2].

If we put $q(z) = z$ in the Th.2 we get the Nunokawa,Obradovic and Owa theorem [1].

PROOF OF THEOREM 1. If

$$\phi = \frac{q_0^2}{q'}[\frac{f'}{f^2} - \frac{q'}{q^2}]$$

then ϕ is analytic in U and

$$-\frac{1}{f(z)} + \frac{1}{q(z)} = \frac{1}{q_0^2}\int_0^z q'(\omega)\phi(\omega)d\omega + c.$$

If we put $q(\omega) = \xi$ we get

$$-\frac{1}{f(z)} + \frac{1}{q(z)} = \frac{1}{q_0^2}\int_0^{q(z)} \phi(q^{-1}(\xi))d\xi + c \tag{3}$$

From the condition $q \in B$ and the relation (3) we get

$$\frac{1}{f(z_1)} - \frac{1}{f(z_2)} + \frac{1}{q(z_2)} - \frac{1}{q(z_1)} = \frac{1}{q_0^2}\int_{q(z_1)}^{q(z_2)} \phi(q^{-1}(\xi))d\xi \tag{4}$$

From Schwarz's Lemma and condition (1) we get $|\phi(z)| \leq 1$ in U. Now from the relation (4) we get

$$\left|\frac{1}{f(z_1)} - \frac{1}{f(z_2)} + \frac{1}{q(z_2)} - \frac{1}{q(z_1)}\right| \leq \frac{|q(z_1) - q(z_2)|}{q_0^2} \tag{5}$$

If $f(z_1) = f(z_2)$ then it is obvious that $q(z_1) = q(z_2)$ or $z_1 = z_2$.

PROOF OF THEOREM 2. If we put

$$P(z) = \frac{z^2 f'}{f^2} - \frac{z^2 q'}{q^2}$$

then we get

$$P'(z) = -z\left[\frac{z}{f(z)} - \frac{z}{q(z)}\right]'' \text{ and } \frac{|P'(z)|}{|q'(z)|} \leq \lambda \text{ in } U.$$

From the relation

$$P(z) = \int_0^{q(z)} \frac{P'(q^{-1}(\xi))}{q'(q^{-1}(\xi))} d\xi$$

we get

$$|P(z)| \leq \lambda |q(z)| \text{ in } U.$$

Now, the condition (1) of Th.1 is obvious.

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