APPLICATIONS OF OUTER MEASURES TO SEPARATION PROPERTIES OF LATTICES AND REGULAR OR σ-SMOOTH MEASURES

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ABSTRACT. Associated with a 0-1 measure $\mu \in I(\mathscr{L})$ where $\mathscr{L}$ is a lattice of subsets of X are outer measures μ' and $\tilde{\mu}$; associated with a σ-smooth 0-1 measure $\mu \in I_\sigma(\mathscr{L})$ is an outer measure μ'' or with $\mu \in I_\sigma(\mathscr{L}')$, $\mathscr{L}'$ being the complementary lattice, another outer measure $\tilde{\tilde{\mu}}$. These outer measures and their associated measurable sets are used to establish separation properties on $\mathscr{L}$ and regularity and σ-smoothness of μ. Separation properties between two lattices $\mathscr{L}_1$ and $\mathscr{L}_2$, $\mathscr{L}_1 \subseteq \mathscr{L}_2$, are similarly investigated. Notions of strongly σ-smooth and slightly regular measures are also used.

KEY WORDS AND PHRASES. Normal lattice, semi-separates, separates, complement generated, countably paracompact, countably compact. Regular, σ-smooth, strongly σ-smooth, slightly regular measures; μ^*-measurable sets.

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1. INTRODUCTION.

Let X be a set, $\mathscr{L}$ a lattice of subsets of X such that $\varnothing$ and X belong to $\mathscr{L}$ and $A(\mathscr{L})$ denote the algebra generated by $\mathscr{L}$. The set $I(\mathscr{L})$ consists of all two-valued (zero or one) finitely additive measures on $A(\mathscr{L})$; the set $I_R(\mathscr{L})$ is a subset of $I(\mathscr{L})$ in which a measure μ is $\mathscr{L}$-regular; the set $I_\sigma(\mathscr{L})$ consists of those elements in $I(\mathscr{L})$ which are σ-smooth on $\mathscr{L}$. Analogous definitions hold for $\mathscr{L}'$, the complementary lattice to $\mathscr{L}$.

Associated with $\mu \in I(\mathscr{L})$ and $\mu \in I_\sigma(\mathscr{L})$ are outer measures μ', $\tilde{\mu}$ and μ'', and with $\mu \in I_\sigma(\mathscr{L}')$, $\tilde{\tilde{\mu}}$. These outer measures have been investigated to some extent in [4, 5, 6].

In this paper, we show how these outer measures can be used systematically to establish separation properties between lattices, and also for establishing regularity of measures or the domination of a suitable measure on $\mathscr{L}$ by a regular σ-smooth measure. In order to achieve some of these goals, we must investigate conditions involving the equality of some of these outer measures on various lattices.

In section 2, we introduce the notations and background material needed for the paper and begin our consideration of separation properties. Section 3 is concerned mainly with lattice-topological conditions which will guarantee equality of certain outermeasures and which will yield regularity. Section 4 extends the work of [4] in considering slightly regular measures on $\mathscr{L}$ and their properties. In addition, specific characteristics are given for the various measurable sets associated with the given outer measures. These, in turn, lead to results on measures in $I_R(\mathscr{L}) \cap I_\sigma(\mathscr{L})$, as well as to measures dominated by such regular measures.

Our notations and terminology are consistent with standard usage (see, for example, [1, 3, 8]).

2. BACKGROUND AND NOTATIONS.

In this section we consider certain lattice properties and definitions, as well as notations. We summarize the most important ones that will be used throughout the paper for the reader's convenience. Related matters can be found in [2, 4, 5, 6, 7].

The set $M(\mathscr{L})$ consists of finite and finitely additive (non-negative) measures on $A(\mathscr{L})$; $I(\mathscr{L})$ mentioned in section 1 is a subset of $M(\mathscr{L})$. The set $M_R(\mathscr{L})$ is a subset of $M(\mathscr{L})$ where $\nu \in M_R(\mathscr{L})$ is said to be regular on $\mathscr{L}$, or $\mathscr{L}$-regular if and only if $\nu(A) = \sup\{\nu(L) \mid L \subseteq A, \ L \in \mathscr{L}\}$, for $A \in A(\mathscr{L})$; again, $I_R(\mathscr{L}) \subseteq M_R(\mathscr{L})$. We say that μ is a measure on $\mathscr{L}$ instead of on $A(\mathscr{L})$ by convention. Also, if $\mathscr{L}$ is a lattice, $\mathscr{L}' = \{L' : L \in \mathscr{L}\}$ denotes the complementary lattice.

We note that since there exists a one-to-one correspondance between prime $\mathscr{L}$-filters and elements of $I(\mathscr{L})$, and similarly between $\mathscr{L}$-ultrafilters and elements of $I_R(\mathscr{L})$, it follows that:

(1) For any $\mu \in I(\mathscr{L})$, there exists $\rho \in I_R(\mathscr{L})$ such that $\mu \leq \rho$ on $\mathscr{L}$.

(2) For any $\mu \in I(\mathscr{L})$, there exists $\lambda \in I_R(\mathscr{L}')$ such that $\mu \leq \lambda$ on $\mathscr{L}'$.

We next associate two finitely subadditive outer measures μ' and $\tilde{\mu}$ with each $\mu \in M(\mathscr{L})$. Let $\mu \in M(\mathscr{L})$, define for $E \subseteq X$

$$\mu'(E) = \inf\{\mu(L') \mid E \subseteq L', \ L \in \mathscr{L}\} .$$

Then it is easy to see that $\mu'(\varnothing) = 0$, μ' is monotone, μ' is finitely subadditive, and $\mu \leq \mu'$ $(\mathscr{L})$, i.e., $\mu \leq \mu'$ on $\mathscr{L}$, and $\mu = \mu'$ $(\mathscr{L}')$. Finally, $\mu = \mu'$ $(\mathscr{L})$ if and only if $\mu \in M_R(\mathscr{L})$. Dually, for $\mu \in M(\mathscr{L})$, define for $E \subseteq X$

$$\tilde{\mu}(E) = \inf\{\mu(L) \mid E \subseteq L, \ L \in \mathscr{L}\} .$$

$\tilde{\mu}$ is also a finitely subadditive outer measure; $\mu = \tilde{\mu}$ $(\mathscr{L})$; $\mu \leq \tilde{\mu}$ $(\mathscr{L}')$; $\mu = \tilde{\mu}$ $(\mathscr{L}')$ if and only if $\mu \in M_R(\mathscr{L}')$.

In this paper, we will be concerned with the special case of $\mu \in I(\mathscr{L})$ in the above. We list several important facts; details can be found in [4, 5, 6].

A lattice $\mathscr{L}$ is normal if for L_1 and L_2 in $\mathscr{L}$, $L_1 \cap L_2 = \varnothing$, there exist L_3 and L_4 in $\mathscr{L}$ such that $L_1 \subseteq L_3'$, $L_2 \subseteq L_4'$ and $L_3' \cap L_4' = \varnothing$. Equivalently, in terms of measures, $\mathscr{L}$ is normal if and only if for a $\mu \in I(\mathscr{L})$, $\nu_1, \nu_2 \in I_R(\mathscr{L})$, $\mu \leq \nu_1$ $(\mathscr{L})$ and $\mu \leq \nu_2$ $(\mathscr{L})$ imply that $\nu_1 = \nu_2$.

PROPOSITION 2.1. If $\mathscr{L}$ is a normal lattice, and if $\nu \in I(\mathscr{L})$ and $\mu \in I_R(\mathscr{L})$ with $\nu \leq \mu$ $(\mathscr{L})$, then $\nu' = \mu'$ $(\mathscr{L})$. Conversely if a lattice $\mathscr{L}$ has this property, then it is normal.

For lattices $\mathscr{L}_1$ and $\mathscr{L}_2$ of subsets of X, $\mathscr{L}_1 \subseteq \mathscr{L}_2$, $\mathscr{L}_1$ semi-separates $\mathscr{L}_2$ if for $L_1 \in \mathscr{L}_1$, $L_2 \in \mathscr{L}_2$, $L_1 \cap L_2 = \varnothing$, there exists $L_0 \in \mathscr{L}_1$ such that $L_2 \subseteq L_0$ and $L_1 \cap L_0 = \varnothing$. The following fact is less well-known; we give a slightly different proof.

PROPOSITION 2.2. Let $\mathscr{L}_1 \subseteq \mathscr{L}_2$ be lattices of subsets of X, then $\mathscr{L}_1$ semi-separates $\mathscr{L}_2$ if and only if for any $\mu \in I_R(\mathscr{L}_1)$, $\mu' = \tilde{\mu}$ $(\mathscr{L}_2)$.

PROOF. a. For any $L_1 \in \mathscr{L}_1$ with $L_2 \subseteq L_1'$, there exists $\tilde{L}_1 \in \mathscr{L}_1$ such that $L_2 \subseteq \tilde{L}_1 \subseteq L_1'$, $\mu(\tilde{L}_1) \leq \mu(L_1')$, so $\tilde{\mu}(L_2) \leq \mu'(L_2)$. For any $\tilde{L}_1$ with $L_2 \subseteq \tilde{L}_1 \in \mathscr{L}_1 \subseteq \mathscr{L}_2$, , we have $\mu'(L_2) \leq \mu'(\tilde{L}_1) = \mu(\tilde{L}_1)$; so $\mu'(L_2) \leq \tilde{\mu}(L_2)$.

b. Suppose not: there exist $L_2 \in \mathscr{L}_2$ and $L_1 \in \mathscr{L}_1$, such that $L_1 \cap L_2 = \varnothing$, and for any $\tilde{L}_1 \supseteq L_2$ where $\tilde{L}_1 \in \mathscr{L}_1$, $\tilde{L}_1 \cap L_1 \neq \varnothing$. The set $S = \{\tilde{L}_1 \cap L_1 : \tilde{L}_1 \supseteq L_2\}$ consisting of non-

empty sets is non-empty: $X \supseteq L_2$ and $X \cap L_1 \neq \varnothing$, so $X \cap L_1 \in S$. We see that S has finite intersection property, i.e., every finite intersection of sets in S is non-empty: Suppose $\tilde{L}_1 \cap L_1$ and $\hat{L}_1 \cap L_1$ are in S, $(\tilde{L}_1 \cap L_1) \cap (\hat{L}_1 \cap L_1) = (\tilde{L}_1 \cap \hat{L}_1) \cap L_1 \neq \varnothing$ since $\tilde{L}_1 \cap \hat{L}_1 \supseteq L_2$. There exists $\mu \in I_R(\mathscr{L}_1)$ such that $\mu(\tilde{L}_1 \cap L_1) = 1$ for all $\tilde{L}_1 \cap L_1$ in S. Since both $\tilde{L}_1$ and L_1 contain $\tilde{L}_1 \cap L_1$, $\mu(\tilde{L}_1) = \mu(L_1) = 1$ for all $\tilde{L}_1 \supseteq L_2$, the outer measure $\mu'(L_2) = \inf\{\mu(A_1) \mid L_2 \subseteq A_1,\ A_1 \in \mathscr{L}_1\} = 1$ by assumption. But $L_2 \subseteq L_1'$ and $\mu(L_1') = 0$, implying that $\mu'(L_2) = 0$, a contradiction.

Again, let $\mathscr{L}_1 \subseteq \mathscr{L}_2$. We recall that every $\mu \in I_R(\mathscr{L}_1)$ can be extended to a $\nu \in I_R(\mathscr{L}_2)$. Also if $\rho \in I_R(\mathscr{L}_2)$ and if $\mathscr{L}_1$ semi-separates $\mathscr{L}_2$, then $\rho|_{A(\mathscr{L}_1)} \in I_R(\mathscr{L}_1)$.

PROPOSITION 2.3. Suppose $\mathscr{L}_1 \subseteq \mathscr{L}_2$ and $\mu \in I_R(\mathscr{L}_1)$ is extended to $\nu \in I_R(\mathscr{L}_2)$. Then ν is $\mathscr{L}_1$-regular on $\mathscr{L}_2'$ (i.e., for any L_2' in $\mathscr{L}_2$, if $\nu(L_2') = 1$, there exists $\tilde{L}_1 \subseteq L_2'$ with $\nu(\tilde{L}_1) = 1$) if and only if $\nu = \mu'$ ($\mathscr{L}_2$).

PROOF. a. For any $L_2 \in \mathscr{L}_2$, $\nu(L_2) = \nu'(L_2) \leq \mu'(L_2)$ since $\mathscr{L}_1 \subseteq \mathscr{L}_2$ and $\nu = \mu$ ($\mathscr{L}_1$). If $\nu'(L_2) = 0$ for some $L_2 \in \mathscr{L}_2$, then $\nu(L_2) = 0$ by regularity. With $\nu(L_2') = 1$, there is an $L_1 \in \mathscr{L}_1$, $L_1 \subseteq L_2'$ and $\nu(L_1) = 1 = \mu(L_1)$. Now $L_2 \subseteq L_1'$, so $\mu'(L_2) = 0$.

b. Assume that $L_2' \in \mathscr{L}_2$ and $\nu(L_2') = 1$. Then $\nu(L_2) = 0 = \mu'(L_2)$. There is $L_1 \in \mathscr{L}_1$, $L_2 \subseteq L_1'$ and $\mu(L_1') = 0$, so $L_1 \subseteq L_2'$ and $\nu(L_1) = 1$.

That $\nu = \mu'$ ($\mathscr{L}_2$) is equivalent to a separation property. When $\mathscr{L}_1 \subseteq \mathscr{L}_2$, we say that $\mathscr{L}_1$ separates $\mathscr{L}_2$ if for B_1 and B_2 in $\mathscr{L}_2$, $B_1 \cap B_2 = \varnothing$, there exist A_1 and A_2 in $\mathscr{L}_1$ such that $B_1 \subseteq A_1$, $B_2 \subseteq A_2$ and $A_1 \cap A_2 = \varnothing$. If $\mathscr{L}_1 \subseteq \mathscr{L}_2$, $\mathscr{L}_1$ separates $\mathscr{L}_2$ implies that $\mathscr{L}_1$ semi-separates $\mathscr{L}_2$.

PROPOSITION 2.4. The condition that $\mathscr{L}_1$ separates $\mathscr{L}_2$ is equivalent to: For every $\mu \in I_R(\mathscr{L}_1)$ and every extension $\nu \in I_R(\mathscr{L}_2)$, we have (i) $\nu = \mu'$ ($\mathscr{L}_2$) and (ii) $\mu' = \tilde{\mu}$ ($\mathscr{L}_2$).

PROOF. a. We will show that $\mathscr{L}_1$ separates $\mathscr{L}_2$ implies that ν is $\mathscr{L}_1$-regular on $\mathscr{L}_2'$: Suppose that $\nu(L_2') = 1$, $L_2 \in \mathscr{L}_2$, then there exists $\tilde{L}_2 \in \mathscr{L}_2$ such that $\tilde{L}_2 \subseteq L_2'$ and $\nu(\tilde{L}_2) = 1$ since $\nu \in I_R(\mathscr{L}_2)$. There is then $L_1 \in \mathscr{L}_1$ such that $L_2 \subseteq L_1$, $\tilde{L}_2 \subseteq \tilde{L}_1$ and $L_1 \cap \tilde{L}_1 = \varnothing$ by the separation property. Now $\tilde{L}_1 \subseteq L_1' \subseteq L_2'$; $\tilde{L}_2 \subseteq \tilde{L}_1$ and $\nu(\tilde{L}_2) = 1$, therefore $\nu(\tilde{L}_1) = 1$, the measure being defined on $\mathscr{L}_1$. By proposition 2.3, $\nu = \mu'$ ($\mathscr{L}_2$). Since $\mathscr{L}_1$ separates $\mathscr{L}_2$, part (ii) follows from proposition 2.2.

b. The proof that $\mathscr{L}_1$ separates $\mathscr{L}_2$ is analogous to that of part b of proposition 2.2. If $\mathscr{L}_1$ does not separate $\mathscr{L}_2$, then a set with finite intersection property can be constructed to lead to a contradiction.

An immediate application of some of the previous material can be given (see [5, 6] for further applications).

PROPOSITION 2.5. If $I(\mathscr{L}) = I_R(\mathscr{L})$, then $\mathscr{L}$ semi-separates $A(\mathscr{L})$, which is equivalent to $\mathscr{L} = \mathscr{L}'$.

PROOF. If $I(\mathscr{L}) = I_R(\mathscr{L})$, one can show that $I_R(\mathscr{L}) = I_R(\mathscr{L}')$. Then for $\mu \in I_R(\mathscr{L}')$, $A \in A(\mathscr{L})$, $\mu(A) = \inf\{\mu(L) \mid A \subseteq L,\ L \in \mathscr{L}\} = \tilde{\mu}(A)$ and $\mu = \mu'$ on $A(\mathscr{L}') = A(\mathscr{L})$. Therefore, $\mu'(A) = \tilde{\mu}(A)$; by proposition 2.2, $\mathscr{L}$ semi-separates $A(\mathscr{L})$.

We will show that in that case $\mathscr{L} \subseteq \mathscr{L}'$ (therefore $\mathscr{L}' \subseteq \mathscr{L}'' = \mathscr{L}$): Let $L_1 \in \mathscr{L}$, then $L_1 = L_2 \in A(\mathscr{L})$ and $L_1 \cap L_1' = \varnothing$. There exists $L_0 \in \mathscr{L}$ such that $L_1' \subseteq L_0$ and $L_1 \cap L_0 = \varnothing$. We have $\varnothing = L_1 \cap L_1' \subseteq L_1 \cap L_0 = \varnothing$, hence $L_1' = L_0 \in \mathscr{L}$.

We now consider the case where $\mu \in I(\mathscr{L})$ has additional smoothness properties. A measure μ is <u>σ-smooth</u> on $\mathscr{L}$ (or $\mu \in I_\sigma(\mathscr{L})$) if and only if for a sequence $\{A_n \in \mathscr{L}\}$, $A_n \downarrow \varnothing$ implies that $\lim_{n \to \infty} \mu(A_n) = 0$. Define for $\mu \in I_\sigma(\mathscr{L})$, $E \subseteq X$,

$$\mu''(E) = \inf\{\sum_1^\infty \mu(L_i') \mid E \subseteq \bigcup L_i', \ L_i \in \mathscr{L}\}.$$

It follows immediately that μ'' is an outermeasure (countably subadditive). In addition:

(1) $\mu \leq \mu''$ $(\mathscr{L})$.

(2) $\mu'' \leq \mu'$.

(3) $\mu'' \leq \mu$ $(\mathscr{L}')$.

(4) If μ is $\mathscr{L}$-regular and σ-smooth on $\mathscr{L}$ (denoted by $\mu \in I_R^\sigma(\mathscr{L})$), then $\mu = \mu'' = \mu'$ $(\mathscr{L})$ and $\mu'' = \mu' = \mu$ $(\mathscr{L}')$.

Dually, using a covering from $\mathscr{L}$, instead of from $\mathscr{L}'$, we define for $\mu \in I_\sigma(\mathscr{L}')$, $E \subseteq X$,

$$\tilde{\tilde{\mu}}(E) = \inf\{\sum_{i=1}^\infty \mu(L_i) \mid E \subseteq \bigcup_1^\infty L_i, \ L_i \in \mathscr{L}\}.$$

Then we have:

(1) $\tilde{\tilde{\mu}} \leq \mu$ $(\mathscr{L})$.

(2) If $\mu \in I_\sigma(\mathscr{L}')$, then $\mu \leq \tilde{\tilde{\mu}}$ $(\mathscr{L}')$.

3. SOME TOPOLOGICAL-TYPE CONDITIONS ON $\mathscr{L}$ FOR $\mu' = \mu''$ ON $\mathscr{L}'$ OR $\mathscr{L}$.

We will first see that when $\mathscr{L}$ is complement generated, the σ-smoothness on $\mathscr{L}'$ of a measure μ will yield $\mathscr{L}$-regularity. Then we will introduce a stronger version of σ-smoothness so that we may obtain $\mathscr{L}$-regularity from the strong σ-smoothness on $\mathscr{L}$ and some topological conditions on $\mathscr{L}$. This strong σ-smoothness will also yield the equality of the two outer measures μ' and μ'' on $\mathscr{L}'$.

A lattice $\mathscr{L}$ is <u>complement generated</u> if for any $L \in \mathscr{L}$, there exists a sequence $\{L_n \in \mathscr{L}\}$ such that $L = \bigcap_{n=1}^\infty L_n'$. A lattice $\mathscr{L}$ is <u>countably paracompact</u> if for any sequence $\{A_n \in \mathscr{L}\}$, $A_n \downarrow \varnothing$, there exists $\{L_n \in \mathscr{L}\}$ such that $A_n \subseteq L_n'$ and $L_n' \downarrow \varnothing$. If $\mathscr{L}$ is complement generated, it is countably paracompact.

NOTE. If $\mathscr{L}$ is countably paracompact, then $I_\sigma(\mathscr{L}') \subseteq I_\sigma(\mathscr{L})$.

It is our aim throughout the paper to give consistently applications of the outermeasures that we have introduced. The following is well-known (see [4, 5, 6]), but we give an alternate proof using outermeasures.

PROPOSITION 3.1. If $\mathscr{L}$ is complement generated and $\mu \in I_\sigma(\mathscr{L}')$, then $\mu \in I_R^\sigma(\mathscr{L})$.

PROOF. $\mathscr{L}$ is countably paracompact and therefore $\mu \in I_\sigma(\mathscr{L})$ by the above note. We have $\tilde{\tilde{\mu}} \leq \mu \leq \mu'' \leq \mu'$ $(\mathscr{L})$. Suppose that for some $L \in \mathscr{L}$, $\tilde{\tilde{\mu}}(L) = 0$ and $\mu'(L) = 1$. Let $L = \bigcap_{n=1}^\infty L_n'$ where $L_n \in \mathscr{L}$ for all n. Then $\mu(L_n') = 1$ for all n, otherwise $\mu(L_N') = 0$ for some N would imply that $\mu'(L) \leq \mu'(L_N) = 0$. Now $L' = \bigcup_n L_n$ and $\mu(L_n) = 0$ for all n means that $\tilde{\tilde{\mu}}(L') = 0$. Then $\tilde{\tilde{\mu}}(X) = 0$, contradicting that $\mu \in I_\sigma(\mathscr{L}')$. Therefore, $\tilde{\tilde{\mu}} = \mu'$ $(\mathscr{L})$ and $\mu = \mu'$ $(\mathscr{L})$; so $\mu \in I_R^\sigma(\mathscr{L})$.

We consider two more notions of σ-smoothness, digress to examine their properties and make some comparisons. A measure $\mu \in I(\mathscr{L})$ is <u>strongly σ-smooth on $\mathscr{L}$</u> or $\mu \in \mathfrak{I}(\mathscr{L})$ if and only if for any sequence $\{L_n \in \mathscr{L}\}$, $L_n \downarrow$, if $\bigcap_n L_n \in \mathscr{L}$ then $\mu(\bigcap_n L_n) = \inf_n \mu(L_n) = \lim_{n \to \infty} \mu(L_n)$. A measure $\mu \in I(\mathscr{L})$ is <u>σ-smooth on $A(\mathscr{L})$</u> or $\mu \in I^\sigma(\mathscr{L})$ if and only if for any sequence $\{A_n \in A(\mathscr{L})\}$, if $A_n \downarrow \varnothing$, then $\lim_{n \to \infty} \mu(A_n) = 0$.

Notes on σ-smoothness:

(1) $\mu \in I''(\mathscr{L})$ if and only if μ is countably additive on $\mathbf{A}(\mathscr{L})$.

(2) $I''(\mathscr{L}) \subseteq \bar{\Phi} \, (\mathscr{L}) \subseteq I_\sigma(\mathscr{L})$.

(3) If $\mu \in I_R^\sigma(\mathscr{L})$, i.e., μ is $\mathscr{L}$-regular and σ-smooth on $\mathscr{L}$, then $\mu \in I''(\mathscr{L})$, i.e., it is σ-smooth on $\mathbf{A}(\mathscr{L})$.

(4) a. If $\mu \in I(\mathscr{L})$ is countably subadditive on $\mathscr{L}'$ (i.e., for all sequences $\{L_n' \in \mathscr{L}'\}$,

$\mu(\bigcup_n L_n') \leq \sum_n \mu(L_n')$) and $\bigcup_n L_n' \in \mathscr{L}'$, then $\mu \in \bar{\Phi} \, (\mathscr{L})$.

b. If $\mu \in \bar{\Phi} \, (\mathscr{L})$, then μ is countably subadditive on $\mathscr{L}'$.

Hence, countably subadditivity on $\mathscr{L}'$ is equivalent to strongly σ-smoothness on $\mathscr{L}$.

PROOF. a. Suppose $L_n \downarrow \bigcap L_n = L \in \mathscr{L}$ and $\mu(L_n) = 1$. By assumption, $\mu(\bigcup_n L_n') \leq \sum_n \mu(L_n') = 0$; so $\mu(L') = 0$. Therefore, $\mu(L) = 1$ and $\mu(\bigcap_n L_n) = \inf_n \mu(L_n)$.

b. Suppose that for $\{L_n' \in \mathscr{L}'\}$, $\mu(\bigcup_n L_n') = 1$, then $\mu(\bigcap L_n) = 0$. We may assume that $L_n \downarrow$, then $\lim_{n \to \infty} \mu(L_n) = 0$ since $\mu \in \bar{\Phi} \, (\mathscr{L})$. As a result, $\mu(L_i') = 1$ from some N on, so the inequality holds.

(5) a. If $\mu \in I_\sigma(\mathscr{L})$ and $\mu'' = \mu' \, (\mathscr{L}')$, then $\mu \in \bar{\Phi} \, (\mathscr{L})$.

b. If $\mu \in \bar{\Phi} \, (\mathscr{L})$, then $\mu'' = \mu' \, (\mathscr{L}')$.

PROOF. See [4].

Dually, we have:

(6) a. If $\mu \in I_\sigma(\mathscr{L}')$ and $\tilde{\tilde{\mu}} = \mu \, (\mathscr{L})$, then $\mu \in \bar{\Phi} \, (\mathscr{L}')$.

b. If $\mu \in \bar{\Phi} \, (\mathscr{L}')$, then $\tilde{\tilde{\mu}} = \mu \, (\mathscr{L})$.

Again, the following proposition is known, but we give an alternate proof using outermeasures.

PROPOSITION 3.2. If $\mathscr{L}$ is complement generated and normal, $\mu \in \bar{\Phi} \, (\mathscr{L})$, then $\mu \in I_R(\mathscr{L})$ and therefore $\mu \in I_R^\sigma(\mathscr{L})$.

PROOF. We have $\mu \leq \mu'' \leq \mu' \, (\mathscr{L})$ and $\mu = \mu' \, (\mathscr{L}')$ in general. Since $\mu \in \bar{\Phi} \, (\mathscr{L})$, by Note (5), $\mu'' = \mu' = \mu \, (\mathscr{L}')$. Suppose that for some $L \in \mathscr{L}$, $\mu(L) = 0$ and $\mu'(L) = 1$; let $L = \bigcap_n L_n'$ where $L_n \in \mathscr{L}$. By normality, there exist A_n and B_n in $\mathscr{L}$ such that $L \subseteq A_n' \subseteq B_n \subseteq L_n'$ for all n, so $L = \bigcap_n A_n' = \bigcap_n B_n = \bigcap_n L_n'$. Since $\mu'(L) = 1$, $\mu'(A_n') = 1$ for all n; so $\mu(A_n) = 0$ and $\mu(B_n') = 0$ for all n. But $L' = \bigcup_n B_n'$, so $\mu''(L') = 0$. Also, $\mu(L') = 1$ and $\mu = \mu'' \, (\mathscr{L}')$ implies that $\mu''(L') = 1$, a contradiction.

To summarize, we have equality of the two outer measures μ'' and μ' on $\mathscr{L}'$ and $\mathscr{L}$ if a. $\mathscr{L}$ is complement generated, $\mu \in I_\sigma(\mathscr{L}')$ or b. $\mathscr{L}$ is complement generated and normal, $\mu \in \bar{\Phi} \, (\mathscr{L})$.

Next, we show

PROPOSITION 3.3. If $\mathscr{L}$ is normal, $\mu \in I_\sigma(\mathscr{L})$, $\nu \in I_R(\mathscr{L})$ and $\mu \leq \nu \, (\mathscr{L})$, then $\nu \in I_\sigma(\mathscr{L}')$.

PROOF. Suppose that there is a sequence $\{L_n \in \mathscr{L}\}$, $L_n \downarrow \varnothing$ and $\nu(L_n') = 1$ for all n. Since $\nu \in I_R(\mathscr{L})$, there exists $\tilde{L}_n \in \mathscr{L}$ such that $\tilde{L}_n \subseteq L_n'$ and $\nu(\tilde{L}_n) = 1$ for all n. Using normality and an argument analogous to the one used in the proof of proposition 3.2, we arrive at a contradiction that $\mu \in I_\sigma(\mathscr{L})$.

It is easy to see that if $\mathscr{L}$ is a δ-lattice (i.e., closed under coutable intersections), and if $\mu \in \bar{\Phi} \, (\mathscr{L})$, then $\mu''(E) = \mu'(E)$ for all $E \subseteq X$.

If we only assume that $\mu \in I_\sigma(\mathscr{L})$, then we must impose topological type conditions $\mathscr{L}$ to insure that $\mu'' = \mu' \, (\mathscr{L})$. We have

PROPOSITION 3.4.   Let $\mu \in I_\sigma(\mathcal{L})$, then $\mu'' = \mu'$ $(\mathcal{L})$

(a) if $\mathcal{L}$ is normal and a δ-lattice; or

(b) if $\mathcal{L}$ is normal and countably paracompact; or

(c) if $\mathcal{L}$ is countably compact (i.e., every countable covering of X by elements from $\mathcal{L}'$ has a finite subcovering).

PROOF. a. Suppose $\mu''(H) = 0$ and $\mu'(H) = 1$ for some $H \in \mathcal{L}$. There exists $\{L_n \in \mathcal{L}\}$, $H \subseteq \underset{n}{\cup} L_n'$, $L_n' \uparrow$, $\mu(L_n') = 0$ for all n. Since $\mathcal{L}$ is a δ-lattice, $\underset{n}{\cup} L_n' = L'$ where $L = \underset{n}{\cap} L_n \in \mathcal{L}$. Using normality analogous to the proof of proposition 3.2, we arrive at a contradiction that μ is σ-smooth on $\mathcal{L}$.

b. As a measure, μ is dominated by some $\nu \in I_R(\mathcal{L})$, $\mu \leq \nu$ $(\mathcal{L})$ or $\nu \leq \mu$ $(\mathcal{L}')$. Since $\mathcal{L}$ is normal, by proposition 3.3, $\nu \in I_\sigma(\mathcal{L}')$; therefore, $\nu \in I_\sigma(\mathcal{L})$ because $\mathcal{L}$ is countably paracompact. With $\nu \in I_R^\sigma(\mathcal{L})$, $\nu = \nu'' = \nu'$ $(\mathcal{L})$. We also have $\nu'' \leq \mu''$ $(\mathcal{L})$. Combining the above, we have $\mu \leq \nu = \nu' = \nu'' \leq \mu'' \leq \mu'$ $(\mathcal{L})$. Since $\mathcal{L}$ is normal, $\nu \in I_R(\mathcal{L})$ with $\mu \leq \nu$ $(\mathcal{L})$, we have $\nu' = \mu'$ $(\mathcal{L})$. Therefore, $\nu' = \mu'' = \mu'$ $(\mathcal{L})$.

c. $\mathcal{L}$ is countably compact if and only if $I(\mathcal{L}) = I_\sigma(\mathcal{L})$. If for some $H \in \mathcal{L}$, $\mu''(H) = 0$, then using a finite cover for X yields a cover for H in $\mathcal{L}'$ with measure zero, so $\mu'(H) = 0$.

Finally, we introduce the following topological-like concept relating two lattices.

DEFINITION 3.1. The lattice $\mathcal{L}_1$ is a sublattice of $\mathcal{L}_2$, $\mathcal{L}_2$ is said to be $\underline{\mathcal{L}_1 \text{ countably compact}}$ if whenever $L_2 \in \mathcal{L}_2$, $L_2 \subseteq \underset{n}{\cup} A_n'$ where $\{A_n \in \mathcal{L}_1\}$, we have $L_2 \subseteq \underset{\text{fin}}{\cup} A_n'$.

NOTE. If $\mathcal{L}_2$ is $\mathcal{L}_1$ countably compact, then $\mathcal{L}_1$ is countably compact because $X \in \mathcal{L}_2$.

PROPOSITION 3.5. (a) If $\mu \in I_\sigma(\mathcal{L}_1)$, $\mathcal{L}_2$ is $\mathcal{L}_1$ countably compact (so $I(\mathcal{L}_1) = I_\sigma(\mathcal{L}_1)$), then $\mu' = \mu''$ $(\mathcal{L}_2)$.

(b) If $\mathcal{L}_2$ is $\mathcal{L}_1$ countably compact, $\mathcal{L}_1$ is complement generated and normal (so $I_R(\mathcal{L}_1) = I_R^\sigma(\mathcal{L}_1)$), and $\mu \in I_R(\mathcal{L}_1)$, then $\mu' = \bar{\mu}$ $(\mathcal{L}_2)$ and $\mathcal{L}_1$ semi-separates $\mathcal{L}_2$.

PROOF. a. For $\mu \in I_\sigma(\mathcal{L}_1)$, $\mu''(E) \leq \mu'(E)$ for all $E \subseteq X$. Equality follows because there is a finite cover for any covering in the definition of $\mu''(E)$.

b. In the proof of proposition 2.2a, we see that $\mu'(L_2) \leq \bar{\mu}(L_2)$ for any $L_2 \in \mathcal{L}_2$ because $\mu = \mu'$ $(\mathcal{L}_1)$ when $\mu \in I_R(\mathcal{L}_1)$. Now if $\mu'(E) = 0$ for some $E \in \mathcal{L}_2$, then $E \subseteq A'$ for some $A \in \mathcal{L}_1$ and $\mu(A') = 0$. Since $A = \underset{n}{\cap} L_n' \subseteq L_n' \in \mathcal{L}_1'$, we may use normality to arrive at $\bar{\mu}(E) = 0$. Therefore, $\mu' = \bar{\mu}$ $(\mathcal{L}_2)$ and by proposition 2.2, $\mathcal{L}_1$ semi-separates $\mathcal{L}_2$.

4. SLIGHTLY REGULAR MEASURES AND OUTER MEASURABLE SETS.

To continue investigating the relationship between the two outer measures, μ' and μ'', we consider first the notion of a slightly regular measure introduced in [4]. We review some of its properties and extend the work done in [4, 5, 6, 7].

DEFINITION 4.1: A σ-smooth measure $\mu \in I_\sigma(\mathcal{L})$ is said to be $\underline{\text{slightly regular on }} \mathcal{L}$, $\mu \in I_s(\mathcal{L})$, if $\mu(L') = 1$ for some $L \in \mathcal{L}$ implies that there exists a sequence $\{L_n \in \mathcal{L}\}$ such that $L' \supseteq \underset{n}{\cap} L_n$ and $\mu(L_n) = 1$ for all n.

$\underline{\text{Properties of }} \mu \in I_s(\mathcal{L})$:

(1) $\mu \in I_s(\mathcal{L})$ if and only if $\mu = \mu''$ $(\mathcal{L})$.

PROOF. a. Suppose $\mu \in I_s(\mathcal{L})$. In general, $\mu \leq \mu''$ $(\mathcal{L})$. Now if there exists $L \in \mathcal{L}$ with

$\mu(L) = 0$, then $\mu(L') = 1$ and there is a sequence $\{L_n \in \mathscr{L}\}$ such that $L' \supseteq \cap L_n$ and $\mu(L_n) = 1$ for all n. Then $\mu''(L) = 0$.

b. Suppose $\mu = \mu''$ $(\mathscr{L})$ and $\mu(A') = 1$ for some $A \in \mathscr{L}$. Then $\mu''(A) = \mu(A) = 0$ and from the definition of μ'', we arrive at $\mu \in I_s(\mathscr{L})$.

For $\mu \in I_o(\mathscr{L})$, μ'' on the intersection of a sequence of sets in $\mathscr{L}$, each with measure 1, is 1. The converse is true if $\mu \in I_s(\mathscr{L})$:

(2) If $\mu \in I_s(\mathscr{L})$ and $\mu''(\underset{n}{\cap} L_n) = 1$ with $L_n \in \mathscr{L}$ for all n, then $\mu(L_n) = 1$ for all n.

PROOF. If $\mu''(\underset{n}{\cap} L_n) = 1$, then $\mu''(L_n) = 1$ for all n. So $\mu(L_n) = \mu''(L_n) = 1$ for all n.

(3) If $\mu \in I_R^\sigma (\mathscr{L})$, then $\mu \in I_s(\mathscr{L})$.

(4) If $\mu \in I_s(\mathscr{L})$, then $\mu \in \overline{I} (\mathscr{L})$.

PROOF. Suppose there exists a sequence $\{L_n \in \mathscr{L}\}$, $L_n \downarrow \underset{n}{\cap} L_n = L \in \mathscr{L}$, $\mu(L_n) = 1$ for all n but $\mu(L) = 0$. Since $\mu(L') = 1$, there exists $\{A_n \in \mathscr{L}\}$ such that $L \subseteq \underset{n}{\cup} A_n'$ and $\mu(A_n') = 0$ for all n; $\mu''(L) = 0$. Also, $L' = \underset{n}{\cup} L_n'$ and $\mu(L_n') = 0$ for all n; $\mu''(L') = 0$. Hence $\mu'' \equiv 0$, contradicting that $\mu \in I_o(\mathscr{L})$.

·(5) If $\mathscr{L}$ semi-separates $\delta(\mathscr{L})$, where $\delta(\mathscr{L})$ is the lattice of all countable intersections of sets from $\mathscr{L}$, and if $\mu \in I_s(\mathscr{L})$, then $\mu \in I_R^\sigma (\mathscr{L})$.

To see the relationship between strongly σ-smoothness and slightly regularity, recall that in the classical case, for an outer measure μ^*, a set $E \subseteq X$ is said to be $\underline{\mu^*\text{-measurable}}$ if for any $A \subseteq X$, $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E')$ (E splits all sets additively with respect to μ^*). Here we will call the sets of all μ'-measurable and μ''-measurable sets $S_{\mu'}$ and $S_{\mu''}$ respectively. As in the classical case, $S_{\mu''}$ is closed under complement and countable union, while $S_{\mu'}$ is an algebra; any set with outer measurable zero is measurable. The restriction of μ' on $S_{\mu'}$ is finitely additive and the restriction of μ'' on $S_{\mu''}$ is countably additive. Recall also that an outer measure μ^* is said to be $\underline{regular}$ if for all $A \subseteq X$, there is a μ^*-measurable set $E \supseteq A$ such that $\mu^*(E) = \mu^*(A)$; E is called a $\underline{\text{measurable cover}}$ for A. There is a duality between statements on μ', μ'' and those on $\tilde{\mu}$ and $\tilde{\tilde{\mu}}$.

$\underline{\text{Remarks}}$:

(1) For $\mu \in I(\mathscr{L})$, $S_{\mu'} = \{E \subseteq X :$ there exists $L \in \mathscr{L}$ such that $E \supseteq L$ with $\mu(L) = 1$ or such that $E' \supseteq L$ with $\mu(L) = 1\}$ and $S_{\tilde{\mu}} = \{E \subseteq X :$ there exists $L \in \mathscr{L}$ such that $E \supseteq L'$ with $\mu(L') = 1$ or such that $E' \supseteq L'$ with $\mu(L') = 1\}$.

PROOF. a. Suppose $E \in S_{\mu'}$. If $\mu'(E) = 0$, we have $L \in \mathscr{L}$ such that $E \subseteq L'$ and $\mu(L') = 0$, or $L \subseteq E'$ and $\mu(L) = 1$. If $\mu'(E) = 1$, since $\mu'(X) = \mu'(E) + \mu'(E')$, we have $\mu'(E') = 0$. Again, there exists $L \in \mathscr{L}$ such that $E' \subseteq L'$ and $\mu(L') = 0$, or $L \subseteq E$ and $\mu(L) = 1$.

b. Let E be in the right-hand side set. If $E \supseteq L \in \mathscr{L}$ with $\mu(L) = 1$, then $E' \subseteq L'$ and $\mu(L') = 0$. So $\mu'(E') = 0$ and $E' \in S_{\mu'}$; therefore $E \in S_{\mu'}$. If $E' \supseteq L \in \mathscr{L}$ with $\mu(L) = 1$, then $\mu'(E) = 0$ and $E \in S_{\mu'}$.

The second part of the statement follows by replacing $\mathscr{L}$ by $\mathscr{L}'$.

(2) For $\mu \in I_o(\mathscr{L})$, $S_{\mu''} = \{E \subseteq X :$ there exists $\{L_n \in \mathscr{L}\}$ such that $E \supseteq \underset{n}{\cap} L_n$ with $\mu(L_n) = 1$ for all n or such that $E' \supseteq \underset{n}{\cap} L_n$ with $\mu(L_n) = 1$ for all n$\}$ and $S_{\tilde{\mu}} = \{E \subseteq X :$ there exists $\{L_n \in \mathscr{L}\}$ such that $E \supseteq \underset{n}{\cap} L_n'$ with $\mu(L_n') = 1$ for all n or such that $E' \supseteq \underset{n}{\cap} L_n'$ with $\mu(L_n') = 1$ for all n$\}$.

PROOF. Omitted.

(3) If $\mu \in I(\mathcal{L})$ $[I_\sigma(\mathcal{L})]$, then μ' $[\mu'']$ is a regular outer measure.

In our case of 0-1 measures, to see whether a set is measurable, it suffices to check the equation on the set X:

(4) For $\mu \in I(\mathcal{L})$ and $E \subseteq X$, if $\mu'(X) = \mu'(E) + \mu'(E')$, then $E \in S_{\mu'}$. The statement is true for $\mu \in I_\sigma(\mathcal{L})$ and μ' is replaced by μ'', $S_{\mu'}$ by $S_{\mu''}$.

(5) If $\mu \in I_s(\mathcal{L})$, then $\mu = \mu''_{|A(\mathcal{L})}$ and $\mu \in I^\sigma(\mathcal{L})$.

PROOF. For any $L \in \mathcal{L}$, we have $\mu(X) = \mu(L) + \mu(L')$. Since $\mu \in I_s(\mathcal{L})$, $\mu = \mu''$ $(\mathcal{L})$; $\mu'' \leq \mu$ $(\mathcal{L}')$. The above equation gives $\mu''(X) \geq \mu''(L) + \mu''(L')$. So $L \in S_{\mu''}$; or $\mathcal{L} \subseteq S_{\mu''}$ which is an algebra and therefore contains $A(\mathcal{L})$. The restriction of μ'' is a measure on $S_{\mu''}$ and therefore on $A(\mathcal{L})$. Since $\mu = \mu''$ $(\mathcal{L})$, $\mu = \mu''_{|A(\mathcal{L})}$. The latter is countably additive, so $\mu \in I^\sigma(\mathcal{L})$.

A series of statements using measurable sets lead to an alternate proof of proposition 3.2:

PROPOSITION 4.1. (a) Suppose $\mu \in I_\sigma(\mathcal{L})$ where $\mathcal{L}$ is normal. If $A \in \mathcal{L}$ and $A = \bigcap_n B_n'$ where $B_n \in \mathcal{L}$, then $A \in S_{\mu''}$.

(b) Suppose $\mu \in \bar{\mathfrak{T}}(\mathcal{L})$ where $\mathcal{L}$ is normal and countably paracompact. If $A \in \mathcal{L}$ and $A = \bigcap_n B_n'$ where $B_n \in \mathcal{L}$, then $A \in S_{\mu'}$.

(c) If $\mu \in \bar{\mathfrak{T}}(\mathcal{L})$ where $\mathcal{L}$ is normal and complement generated, then $\mu \in I_R^\sigma(\mathcal{L})$. [proposition 3.2.]

PROOF. a. Since $A \subseteq B_n'$ and $\mathcal{L}$ is normal, there exists C_n and D_n in $\mathcal{L}$ such that $A \subseteq C_n' \subseteq D_n \subseteq B_n'$ for all n. So we have $A = \bigcap_n C_n' = \bigcap_n D_n = \bigcap_n B_n'$. Sets of μ''-measure zero are in $S_{\mu''}$. If $\mu''(A) = 1$, then $\mu''(C_n') = 1$; Since $\mu'' \leq \mu$ $(\mathcal{L}')$, $\mu(C_n') = 1$ and therefore $\mu(D_n) = 1$ for all n. Now $A = \bigcap_n D_n$ and $\mu(D_n) = 1$ for all n, so $A \in S_{\mu''}$.

b. By normality and countably paracompactness, $\mu' = \mu''$ $(\mathcal{L})$. If $\mu'(A) = 1$, then $\mu''(A) = 1$ and $A \in S_{\mu''}$ by part a. In this case $S_{\mu''} \cap \mathcal{L} = S_{\mu'} \cap \mathcal{L}$, so $A \in S_{\mu'}$.

c. We have therefore $\mathcal{L}$ is countably paracompact and normal and $\mathcal{L} \subseteq S_{\mu'}$. Since $S_{\mu'} \cap \mathcal{L} = \{L \in \mathcal{L} : \mu'(L) = \mu(L)\}$, so $\mu'(L) = \mu(L)$ for all $L \in \mathcal{L}$, or $\mu = \mu'$ $(\mathcal{L})$, or $\mu \in I_R^\sigma(\mathcal{L})$.

Using the regularity of the outer measures, we give

ALTERNATE PROOF FOR PROPOSITION 3.2. If $\mathcal{L}$ is normal, $\mu \in I_\sigma(\mathcal{L})$, $\nu \in I_R(\mathcal{L})$ and $\mu \leq \nu$ $(\mathcal{L})$, then $\nu \in I_\sigma(\mathcal{L}')$.

PROOF. We have $\mu \leq \nu = \nu' \leq \mu'$ $(\mathcal{L})$ and $\mu' = \nu'$ $(\mathcal{L})$ by normality. Therefore, $\mu \leq \mu'' \leq \mu' = \nu' = \nu$ $(\mathcal{L})$. Suppose that for $\{L_n \in \mathcal{L}\}$, $L_n' \downarrow \varnothing$. Then $L_n \uparrow X$ or $X = \bigcup_n L_n$ and $1 = \mu''(X) = \lim_{n \to \infty} \mu''(L_n)$ by the regularity of μ''. So there exists N such that $\mu''(L_N) = 1$ and therefore $\nu(L_n) = 1$ for $n \geq N$, or $\nu(L_n') = 0$ for $n \geq N$. Hence $\nu \in I_\sigma(\mathcal{L}')$.

PROPOSITION 4.2. If $\mathcal{L}$ is normal and a δ-lattice, $\mu \in I_\sigma(\mathcal{L})$, $\nu \in I_R(\mathcal{L})$ and $\mu \leq \nu$ $(\mathcal{L})$, then $\nu \in \bar{\mathfrak{T}}(\mathcal{L}')$.

PROOF. By proposition 3.4a and normality, we have $\mu \leq \mu'' = \mu' = \nu' = \nu$ $(\mathcal{L})$. Suppose that for $\{L_n \in \mathcal{L}\}$ $L_n' \downarrow L' \in \mathcal{L}'$. Then $L_n \uparrow L$ and $\mu''(L_n) \uparrow \mu''(L)$, or $\nu(L_n) \uparrow \nu(L)$, $\nu(L_n') \downarrow \nu(L')$. So $\nu \in \bar{\mathfrak{T}}(\mathcal{L}')$.

PROPOSITION 4.3. If $\mu \in \bar{\mathfrak{T}}(\mathcal{L})$ and $\mathcal{L} \subseteq S_{\mu'}$, then $\mu = \mu''$ $(\mathcal{L})$ and therefore $\mu \in I_s(\mathcal{L})$.

PROOF. In general, $\mu \leq \mu''$ $(\mathcal{L})$. Suppose that for some $L \in \mathcal{L}$, $\mu(L) = 0$ and $\mu''(L) = 1$.

Since $\mu \in \bar{\mathfrak{I}}(\mathscr{L})$, $\mu''(L') = \mu'(L') = \mu(L') = 1$. But L' is μ''-measurable, so $\mu''(L) = 0$, a contradiction. Therefore, $\mu = \mu''$ $(\mathscr{L})$ or $\mu \in I_s(\mathscr{L})$.

It follows immediately that

COROLLARY. If $\mu \in \bar{\mathfrak{I}}(\mathscr{L}')$ and $\mathscr{L} \subseteq S_{\tilde{\mu}}$, then $\mu \in I^\sigma(\mathscr{L})$.

PROPOSITION 4.4. If $\mu \in I_s(\mathscr{L})$, $\nu \in I_\sigma(\mathscr{L})$ and $\mu \leq \nu$ $(\mathscr{L})$, then $\nu \in I_s(\mathscr{L})$.

PROOF. In general, $\nu \leq \nu''$ $(\mathscr{L})$. By the definition of the outer measures, $\mu \leq \nu$ $(\mathscr{L})$ implies that $\nu'' \leq \mu''$ $(\mathscr{L})$. Combining, we have $\mu \leq \nu \leq \nu'' \leq \mu''$ $(\mathscr{L})$. Since $\mu \in I_s(\mathscr{L})$, $\mu = \mu''$ $(\mathscr{L})$; therefore $\nu = \nu''$ $(\mathscr{L})$ or $\nu \in I_s(\mathscr{L})$.

Notes on sets in $\mathscr{L}$ which are measurable:

(1) If $\mu \in I(\mathscr{L})$, then $S_{\mu'} \cap \mathscr{L} = \{L \in \mathscr{L} : \mu(L) = \mu'(L)\}$.

(2) For $\mu \in I_\sigma(\mathscr{L})$, $\mu \in \bar{\mathfrak{I}}(\mathscr{L})$ if and only if $S_{\mu''} \cap \mathscr{L} = \{L \in \mathscr{L} : \mu(L) = \mu''(L)\}$.

PROOF. a. Suppose $\mu \in \bar{\mathfrak{I}}(\mathscr{L})$ and $L \in S_{\mu''} \cap \mathscr{L}$. Using Note 5b on σ-smoothness, we get $\mu''(X) = \mu''(L) + \mu''(L') = \mu''(L) + \mu(L')$; while $\mu''(X) = \mu(X) = \mu(L) + \mu(L')$. Therefore $\mu''(L) = \mu(L)$.

b. Suppose $\mu \in \bar{\mathfrak{I}}(\mathscr{L})$ and $L \in \mathscr{L}$, $\mu''(L) = \mu(L)$. Then $\mu''(X) = \mu(X) = \mu(L) + \mu(L') = \mu''(L) + \mu''(L')$, hence $L \in S_{\mu''}$.

c. Assume that a set in $\mathscr{L}$ is μ''-measurable if and only if μ'' and μ have the same value on it. We will show that $\mu'' = \mu'$ $(\mathscr{L})$. In general, $\mu'' \leq \mu = \mu'$ $(\mathscr{L})$. Suppose that for some $L \in \mathscr{L}$, $\mu''(L') = 0$ and $\mu(L') = 1$, then $\mu''(L) = 1$ and $\mu(L) = 0$. We have $\mu''(X) = 1 = \mu''(L') + \mu''(L)$; so $L \in S_{\mu''}$. We have $\mu(L) = 0$ and $\mu''(L) = 1$, a contradiction.

Recall that $\mu \in I_s(\mathscr{L})$ if and only if for any $L \in \mathscr{L}$, $\mu(L') = 1$ implies that there is a sequence $\{L_n \in \mathscr{L}\}$ such that $L' \supseteq \bigcap_n L_n$ and $\mu(L_n) = 1$ for all n. By contrast, such a condition for $\mu''(L') = 1$ is equivalent to $\mathscr{L} \subseteq S_{\mu''}$:

(3) For $\mu \in I_\sigma(\mathscr{L})$, $\mathscr{L} \subseteq S_{\mu''}$ if and only if $\mu''(L') = 1$ for any $L \in \mathscr{L}$ implies that there is a sequence $\{L_n \in \mathscr{L}\}$ such that $L' \supseteq \bigcap_n L_n$ and $\mu(L_n) = 1$ for all n.

PROOF. a. Suppose $\mu''(L') = 1$ for some $L \in \mathscr{L} \subseteq S_{\mu''}$. Then $\mu''(L) = 0$ and we obtain a sequence $\{L_n \in \mathscr{L}\}$ such that $L' \supseteq \bigcap_n L_n$ and $\mu(L_n) = 1$ for all n.

b. Let $L \in \mathscr{L}$. If $\mu''(L') = 0$, then $L' \in S_{\mu''}$ and therefore $L \in S_{\mu''}$. If $\mu''(L') = 1$, then $L' \supseteq \bigcap_n L_n$ for some $\{L_n \in \mathscr{L}\}$ and $\mu(L_n) = 1$ for all n. Hence $L \in S_{\mu''}$ by Note (2).

A dual statement for $\tilde{\mu}$ follows immediately if we replace $\mathscr{L}$ by $\mathscr{L}'$.

(3') For $\mu \in I_\sigma(\mathscr{L}')$, $\mathscr{L} \subseteq S_{\tilde{\mu}}$ if and only if $\tilde{\mu}(L) = 1$ for any $L \in \mathscr{L}$ implies that there is a sequence $\{L_n \in \mathscr{L}\}$ such that $L \supseteq \bigcap_n L_n'$ and $\mu(L_n') = 1$ for all n.

A slightly stronger condition implies that $\mathscr{L} \subseteq S_{\mu''}$:

(4) For $\mu \in I_\sigma(\mathscr{L})$, the condition that if $\mu''(L') = 1$ for some $L \in \mathscr{L}$ then there exists $\tilde{L} \in \mathscr{L}$ such that $L' \supseteq \tilde{L}$ and $\mu(\tilde{L}) = 1$ implies that $\mathscr{L} \subseteq S_{\mu''}$. Furthermore, $\mu \leq \mu''|_{\mathbb{A}(\mathscr{L})}$ $(\mathscr{L})$ and $\mu''|_{\mathbb{A}(\mathscr{L})} \in I_R^\sigma(\mathscr{L})$.

PROOF. a. If $\mu''(L') = 0$ for some $L \in \mathscr{L}$, then L' and therefore L is μ''-measurable. If $\mu''(L') = 1$, then there exists $\tilde{L} \in \mathscr{L}$ such that $L' \supseteq \tilde{L}$ and $\mu(\tilde{L}) = 1$. Take $L_n = \tilde{L}$ for all n, then by Note (3) above, $L \in S_{\mu''}$.

b. In general, we have $\mu \leq \mu''$ $(\mathscr{P})$. We know that $\mu''{\mid}_{A(\mathscr{L})}$ is a measure on $A(\mathscr{L})$ because $\mathscr{P} \subseteq S_{\mu''}$ implies that $A(\mathscr{P}) \subseteq S_{\mu''}$ and μ'' is a measure on $S_{\mu''}$. In this case $\mu''{\mid}_{A(\mathscr{L})} \in I_R^\sigma(\mathscr{L})$: if $\mu''(L') = 1$ for some $L' \in \mathscr{P}'$, then there exists $\tilde{L} \subseteq L'$ and $\mu''(\tilde{L}) \geq \mu(\tilde{L}) = 1$.

(5) If $\mathscr{P}$ is complement generated, then $\mathscr{P} \subseteq S_{\tilde{\tilde{\mu}}}$.

PROOF. For $L = \underset{n}{\cap} L_n'$, if $\tilde{\tilde{\mu}}(L) = \tilde{\tilde{\mu}}(\underset{n}{\cap} L_n') = 1$, then $\tilde{\tilde{\mu}}(L_n') = 1$ for all n; so $L \in S_{\tilde{\tilde{\mu}}}$.

When $\mathscr{P}$ semi-separates $\delta(\mathscr{P})$ and $\mathscr{P} \subseteq S_{\mu''}$, we will get $\mathscr{P}$-regularity for a. $\mu \in \overline{\mathfrak{I}}(\mathscr{L})$ or b. $\mu''{\mid}_{A(\mathscr{L})}$ associated with $\mu \in I_\sigma(\mathscr{P})$.

PROPOSITION 4.5. If $\mu \in \overline{\mathfrak{I}}(\mathscr{L})$, $\mathscr{P}$ semi-separates $\delta(\mathscr{P})$ and $\mathscr{P} \subseteq S_{\mu''}$, then $\mu \in I_R^\sigma(\mathscr{L})$.

PROOF. Suppose $A \in \mathscr{P}$ and $\mu(A') = 1$. Since $\mu \in \overline{\mathfrak{I}}(\mathscr{L})$ and $\mathscr{P} \subseteq S_{\mu''}$, $\mu \in I_s(\mathscr{P})$ by proposition 4.3. So there exists $\{L_n \in \mathscr{P}\}$ such that $A' \supseteq \underset{n}{\cap}L_n$ and $\mu(L_n) = 1$ for all n. We have $D = \underset{n}{\cap}L_n \in \delta(\mathscr{P})$ and $D \subseteq A'$: there exists $\tilde{L} \in \mathscr{P}$ such that $D \subseteq \tilde{L} \subseteq A'$ by semi-separateness. It follows from the definition of μ'' that $\mu''(D) = 1$; therefore $1 = \mu''(\tilde{L}) = \mu(\tilde{L})$ since $\tilde{L} \in S_{\mu''}$. So $\mu \in I_R(\mathscr{L})$.

A dual statement for $\tilde{\tilde{\mu}}$ follows:

COROLLARY. If $\mu \in \overline{\mathfrak{I}}(\mathscr{L}')$, $\mathscr{P}$ semi-separates $\delta(\mathscr{L}')$ and $\mathscr{P} \subseteq S_{\tilde{\tilde{\mu}}}$, then $\mu \in I_R^\sigma(\mathscr{L})$.

PROPOSITION 4.6. If $\mu \in I_\sigma(\mathscr{L})$, $\mathscr{P}$ semi-separates $\delta(\mathscr{L})$ and $\mathscr{P} \subseteq S_{\mu''}$, then $\mu \leq \mu''$ $(\mathscr{L})$ and $\mu''{\mid}_{A(\mathscr{L})} \in I_R^\sigma(\mathscr{L})$.

PROOF. We have $\mu \leq \mu''$ $(\mathscr{L})$ and $\mu''{\mid}_{A(\mathscr{L})}$ is a measure by our previous argument. To see that $\mu''{\mid}_{A(\mathscr{L})} \in I_R(\mathscr{L})$, let $\mu''(A') = 1$ for some $A \in \mathscr{L}$. Then by Note 3 above, there exists $\{L_n \in \mathscr{L}\}$ such that $A' \supseteq \underset{n}{\cap}L_n$ and $\mu(L_n) = 1$ for all n. As in the above argument, $D = \underset{n}{\cap}L_n \in \delta(\mathscr{L})$ and $D \subseteq A'$, there exists $\tilde{L} \in \mathscr{L}$ such that $\tilde{L} \subseteq A'$ and $\mu''(\tilde{L}) = 1$.

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