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A NEW CRITERION FOR STARLIKE FUNCTIONS

LING YI
Department of Mathematics
Harbin Institute of Technology
Harbin, P R. China

SHUSEN DING
Department of Mathematics
The Florida State University
Tallahassee, Florida, 32306-3027

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ABSTRACT. In this paper we shall get a new criterion for starlikeness, and the hypothesis of this criterion is much weaker than those in [1] and [2].

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1. INTRODUCTION AND PRELIMINARIES.

Let $\mathcal{A}$ be the class of functions $f(z)$, which are analytic in the unit disc $D = \{z: |z| < 1\}$, with $f(0) = f'(0) - 1 = 0$. Let S be the set of starlike functions, $S = \{f(z) \in \mathcal{A},\ Re(zf'(z)/f(z)) > 0,\ z \in D\}$.

R. Singh and S. Singh in [1] proved that if $f(z) \in \mathcal{A}$ and $Re[f'(z) + zf''(z)] > 0$, $z \in D$, then $f(z) \in S$.

Recently, R. Singh and S. Singh in [2] proved that if $f(z) \in \mathcal{A}$ and $Re[f'(z) + zf''(z)] > -\frac{1}{4}$, $z \in D$, then $f(z) \in S$.

In this paper we shall show that the assertion of R. Singh and S. Singh holds under a much weaker hypothesis.

LEMMA 1. Suppose that the function ψ: $C^2 \times D \to C$ satisfies the condition $Re\psi(ix,\ y;\ z) \le \delta$ for all real $x, y \le -\frac{(1+x^2)}{2}$ and all $z \in D$. If $p(z) = 1 + p_1 z + \cdots$ is analytic in D and

$$Re\psi(p(z),\ zp'(z);\ z) > \delta,\ for\ z \in D$$

then $Re(p(z)) > 0$ in D.

A general form of this lemma can be found in [3]. In [4] the authors got the following result.

LEMMA 2. Let $\alpha > 0$, $\beta < 1$. If the function p is analytic in D, with $p(0) = 1$ and

$$Re[p(z) + \alpha z p'(z)] > \beta, \quad z \in D$$

then $Re(p(z)) > (2\beta - 1) + 2(1-\beta)F(1,\ \frac{1}{\alpha},\ \frac{1}{\alpha} + 1;\ -1)$, $z \in D$, where $F(a,\ b,\ c;\ x)$ is a hypergeometric function. This result is sharp.

By taking $\alpha = 1$ in lemma 2, we obtain

LEMMA 3. Let $\beta < 1$. If the function p is analytic in D, with $p(0) = 1$ and

$$Re[p(z) + zp'(z)] > \beta, \quad z \in D$$

then $Re(p(z)) > (2\beta - 1) + 2(1-\beta)\ln 2$, $z \in D$, and the result is sharp.

2. MAIN RESULT

THEOREM. If $f(z) \in \mathcal{A}$ and

$$Re\,[f'(z) + zf''(z)] > 1 - \frac{3}{4(1 - \ln 2)^2 + 2} \approx -0.263, \quad z \in D \tag{1}$$

then $f(z) \in S$.

PROOF. By using lemma 3, from (1) we have

$$Re(f'(z)) > 1 - \frac{3(1 - \ln 2)}{2(1 - \ln 2)^2 + 1} > 0, \qquad z \in D. \tag{2}$$

From (2) and lemma 3, we have

$$Re\frac{f(z)}{z} > -2 + \frac{3}{2(1 - \ln 2)^2 + 1} \approx 0.526, \qquad z \in D. \tag{3}$$

Now, we let $p(z) = zf'(z)/f(z)$ and $\lambda(z) = f(z)/z$, then $p(z)$ is analytic in D and $p(0) = 1$, $Re\{\lambda(z)\} > -2 + \frac{3}{2(1-\ln 2)^2+1}$. A simple computation shows that

$$f'(z) + zf''(z) = \lambda(z)[p^2(z) + zp'(z)] = \psi(p(z),\ zp'(z);\ z),$$

where $\psi(u,\ v;\ z) = \lambda(z)(u^2 + v)$. Using (1), we have $Re\,[\psi(p(z),\ zp'(z); z)] > 1 - \frac{3}{4(1-\ln 2)^2+2}$ for each $z \in D$. Now for all real x, $y \leq -\frac{1}{2}(1 + x^2)$, we have

$$Re\,[\psi(ix,\ y;\ z)] = (y - x^2)Re\,[\lambda(z)] \leq -\frac{1}{2}(1 + 3x^2)Re\,[\lambda(z)] \leq -\frac{1}{2}Re\,[\lambda(z)] \tag{4}$$

for each $z \in D$. Note that $Re\,[\lambda(z)] > -2 + \frac{3}{2(1-\ln 2)^2+1}$, from (4) we get

$$Re\,[\psi(ix,\ y;\ z)] \leq 1 - \frac{3}{4(1 - \ln 2)^2 + 2}$$

for all $z \in D$. Thus by lemma 1, $Re\,[p(z)] > 0$ in D, that is, $f(z) \in S$.

REMARK. For $\beta < 1$, let $R(\beta) = \{f \in \mathcal{A} : Re[f'(z) + zf''(z)] > \beta,\ z \in D\}$. It was proved in [4] that if $f(z) \in R(\alpha_0)$ ($\alpha_0 = \frac{1-2\ln 2}{2-2\ln 2} \approx -0.61$), then $f(z)$ is univalent, and the constant α_0 can not be replaced by any less one. Our present theorem yields $R\left(1 - \frac{3}{4(1-\ln 2)^2+2}\right) \subset S$. Thus, a natural problem which arises is to find $\inf\{\beta :\ R(\beta) \subset S\}$.

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