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CLOSE-TO-STARLIKE LOGHARMONIC MAPPINGS

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Abstract

We consider logharmonic mappings of the form $f = z|z|^{2\beta}h\overline{g}$ defined on the unit disc U which can be written as the product of a logharmonic mapping with positive real part and a univalent starlike logharmonic mapping. Such mappings will be called close-to-starlike logharmonic mappings. Representation theorems and distortion theorems are obtained. Moreover, we determine the radius of univalence and starlikeness of these mappings.

Key Words and Phrases: Logharmonic mappings, close to starlike, positive real part, radius of starlikeness and univalence.

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1 Introduction

Let $H(U)$ be the linear space of all analytic functions defined on the unit disc $U = \{z; |z| < 1\}$ and let B be the set of all functions $a \in H(U)$ such that $|a(z)| < 1$ for all $z \in H(U)$. A logharmonic mapping is a solution of the non-linear elliptic partial differential equation

$$\overline{f_{\overline{z}}} = a.\frac{\overline{f}}{f}.f_z \tag{1.1}$$

where the second dilatation function a is in B. Observe that nonconstant logharmonic mappings are open and orientation preserving on U. If f does not vanish on U, then f is of the form

$$f = H.\overline{G}$$

where H and G are in $H(U)$. On the other hand, if f vanishes at 0, but has no other zeros in U, then f admits the representation

$$f(z) = z^m|z|^{2\beta m}h(z)\overline{g(z)}$$

where

a) m is nonnegative integer

b) $\beta = \overline{a(0)}(1 + a(0))/(1 - |a(0)|^2)$ and therefore, $Re\ \beta > -1/2$.

c) h and g are analytic in U, $g(0) = 1$ and $h(0) \neq 0$.

If f is a univalent logharmonic mapping on U, then either $0 \notin f(U)$ and $log\ f$ is univalent and harmonic on U or, if $f(0) = 0$, then f is of the form $f = z|z|^{2\beta}h\overline{g}$ where $Re\ \beta > -1/2$ and $0 \notin h.g(U)$ and where $F(\zeta) = log\ f(e^{\zeta})$ is univalent and harmonic on the half-plane $\{\zeta; Re\ \zeta < 0\}$ (for more details see [1]).If in addition, $f(U)$ is starlike domain then F is closely connected with nonparametric minimal surfaces over domains Ω of the form $\Omega = \{w = u + iv : -\infty < u < u_0(v),\ v \in \mathcal{R}\ and\ u_0(v + 2\pi) = u_0(v)\ for\ all\ v \in \mathcal{R}\}$, whose corresponding Gauss mapping is periodic. Indeed, there induces a non-parametric minimal surface $(u, v, s = G(u, v))$ over Ω defined by the defferential relations:

$$\overline{F_{\bar{z}}} = AF_z,\quad (s_z(z))^2 = -A(z)(F_z(z))^2$$

where $A \in B$ such that $A(z + 2\pi i) = A(z)$. For elementary facts concerning minimal surfaces, we refer the reader to [4] and [5].

Let S^*_{Lh} denote the set of all univalent logharmonic mappings f defined on U such that $f(0) = 0$, $h(0) = g(0) = 1$ and such that $f(U)$ is a starlike domain. Also, let $S^* = \{f \in S^*_{Lh}\ and\ f \in H(U)\}$. A detailed study of these mappings can be found in [2]. In particular, the following is a representation theorem for mappings in S^*_{Lh}.

Theorem A [2,Theorem 2.1].

a) If $f = z|z|^{2\beta}h^*\overline{g^*} \in S^*_{Lh}$, *then* $\phi(z) = \dfrac{zh^*}{g^*} \in S^*$.

b) For any given $\phi \in S^*$ *and* $a \in B$, *there are* h^* *and* g^* *in* $H(U)$ *uniquely determined such that*

i) $0 \notin h^*.g^*(U)$; $h^*(0) = g^*(0) = 1$

ii) $\phi(z) = \dfrac{zh^*}{g^*}$

iii) $f(z) = z|z|^{2\beta}h^*(z)\overline{g^*(z)}$ *is a solution of (1.1) in* S^*_{Lh}, *where*

$\beta = \overline{a(0)}(1 + a(0))/(1 - |a(0)|^2)$.

In Section 2, we include representaion theorems and a distortion theorem for logharmonic mappings with positive real part.

In Section 3, we shall deal with close-to-starlike logharmonic mappings. Representation theorems are given. We obtain the radius of starlikeness and univalence of these mappings. Moreover, distortion theorems for close-to-starlike logharmonic mappings are included.

2 Logharmonic mappings with positive real part

Let P_{Lh} be the set of all logharmonic mappings R defined on the unit disk U which are of the form $R = H.\overline{G}$ where H and G are in $H(U)$, $H(0) = G(0) = 1$ and such that $Re\ R(z) > 0$ for all $z \in U$. In particular, the set P of all analytic functions $p(z)$ in U with $p(0) = 1$ and $Re\ p(z) > 0$ in U is a subset of P_{Lh}.

We begin by observing that the set P_{Lh} is logarithmically convex. In other words, for given $\lambda \in (0,1)$ and given functions $R_1(z)$ and $R_2(z)$ in P_{Lh} which are solutions of (1.1) with respect to the same $a \in B$, the mapping $S(z) = R_1(z)^{\lambda}R_2(z)^{1-\lambda}$ belongs also to P_{Lh} and satisfies (1.1) with respect to the same a.

Our first result of this section connects P_{Lh} and P.

Theorem 2.1. *Let $R = H.\overline{G} \in P_{Lh}$. Then $p = H/G \in P$. Conversely, given $p \in P$ and $a \in B$, then there exists nonvanishing functions H and G in $H(U)$ such that $p = H/G$, $R = H.\overline{G} \in P_{Lh}$ and R is a solution of (1.1) with respect to the given a.*

Proof: The first assertion is obvious. Suppose that $p \in P$ and $a \in B$ are given. Define

$$G(z) = exp\left(\int_0^z \frac{a}{1-a}\frac{p'(s)}{p(s)}ds\right). \tag{2.1}$$

Then

$$R(z) = p(z)|G(z)|^2 \tag{2.2}$$

has the desired properties. □

The previous theorem allows us to give an integral representation for mappings in P_{Lh}. Indeed, for $p \in P$, there is a probability measure μ defined on the Borel $\sigma - algebra$ of ∂U such that

$$p(z) = \int_{\partial U} \frac{1+\zeta z}{1-\zeta z} d\mu(\zeta). \tag{2.3}$$

On the other hand, there is for each $a \in B$, a probability measure ν defined on the Borel $\sigma - algebra$ of ∂U such that

$$\frac{a(z)}{1-a(z)} = \frac{1-|a(0)|^2}{|1-a(0)|^2}\int_{\partial U}\frac{\eta z}{1-\eta z}d\nu(\eta) + \frac{a(0)}{1-a(0)}. \tag{2.4}$$

Substituting (2.3) and (2.4) into (2.1) and (2.2), we get

Theorem 2.2. *A function f belongs to the class P_{Lh} if and only if there are two probability measures μ and ν on the Borel sets of ∂U and an $a(0) \in U$ such that*

$$R(z) = \int_{\partial U} \frac{1+\zeta z}{1-\zeta z} d\mu(\zeta) exp\left(2Re\int_0^z K_1(s, a(0))ds\right),$$

where

$$K_1(z, a(0)) = \left[\frac{1-|a(0)|^2}{|1-a(0)|^2}\int_{\partial U}\frac{\eta z}{1-\eta z}d\nu(\eta) + \frac{a(0)}{1-a(0)}\right]\frac{\int_{\partial U}\frac{\zeta z}{(1-\zeta z)^2}d\mu(\zeta)}{\int_{\partial U}\frac{1+\zeta z}{1-\zeta z}d\mu(\zeta)}.$$

As one observes, this integral representation does not look to be a very promising tool to solve extremal problems. However, we shall see in Theorem 2.3 that if $a(0) = 0$, then $\max_{f \in P_{Lh}} |f(z)|$ is attained for $\mu = \nu = \delta_1$, where δ_1 is the Dirac measure concentrated at the point 1. Also, $\min_{f \in P_{Lh}} |f(z)|$ occurs if $\mu = \nu = \delta_{-1}$, where δ_{-1} is the Dirac measure concentrated at -1. Finally, let us observe that $f(z) \in P_{Lh}$ and $|\eta| < 1$ imply that $f(\eta z) \in P_{Lh}$.

Next, we obtain a distortion theorem for the set P_{Lh}.

Theorem 2.3. *Let $R(z) = H(z).\overline{G(z)} \in P_{Lh}$, and suppose that $a(0) = 0$. Then for $z \in U$ we have*

i) $e^{-2|z|/(1-|z|)} \leq |R(z)| \leq e^{2|z|/(1-|z|)}$

ii) $|R_z(z)| \leq \dfrac{2}{(1-|z|)(1-|z|^2)}e^{2|z|/(1-|z|)}$

iii) $|R_{\overline{z}}(z)| \leq \dfrac{2|z|}{(1-|z|)(1-|z|^2)}e^{2|z|/(1-|z|)}$.

Equality occurs for the right hand side inequalities if $R(z)$ is one of the functions of the form

$R_0(\zeta z)$, $|\zeta| = 1$, *where*

$$R_0(z) = \frac{1+z}{1-z}\left|\frac{1-z}{1+z}\right| e^{Re\frac{2z}{1-z}},$$

and for the left hand side inequality if $R(z)$ is one of the functions of the form

$$\frac{1}{R_0(\zeta z)},\ |\zeta| = 1.$$

Proof: i):From Theorem 2.1, it follows that R admits the representation

$$R(z) = p(z)exp\left(2Re\int_0^z \frac{a(s)}{1-a(s)}\frac{p'(s)}{p(s)}ds\right), \tag{2.5}$$

where $a \in B$ and $p \in P$.

Fix $|z| = r$. Then we have

$$|p(z)| \leq \frac{1+r}{1-r}, \tag{2.6}$$

$$\left|\frac{1}{1-a(z)}\right| \leq \frac{1}{1-r} \tag{2.7}$$

and

$$\left|z\frac{p'(z)}{p(z)}\right| \leq \frac{2r}{1-r^2}. \tag{2.8}$$

To see the last inequality, define $b = \frac{p-1}{p+1}$. Then b is a Schwarz function (i.e. $b \in H(U)$, $b(0) = 0$ and $|b| < 1$ on U) and we get

$$\left|\frac{zp'(z)}{p(z)}\right| = r\left|\frac{2b'(z)}{(1-b(z))^2}.\frac{1-b(z)}{1+b(z)}\right| \leq r\frac{2|b'(z)|}{1-|b(z)|^2} \leq \frac{2r}{1-r^2}.$$

Therefore, we obtain

$$|R(z)| \leq \frac{1+r}{1-r}exp\left(2\int_0^r \frac{1}{1-t}\frac{2t}{1-t^2}dt\right) = e^{\frac{2r}{1-r}}.$$

Equality occurs if and only if $a(z) = \zeta z$ and $p(z) = \frac{1+\zeta z}{1-\zeta z}$, $|\zeta| = 1$, which leads to $R(z) = R_0(\zeta z)$.

It remains to show the left hand side inequality. Observe that $R \in P_{Lh}$ implies that $\frac{1}{R} \in P_{Lh}$. Applying the right hand side inequality to the function $\frac{1}{R}$, we obtain

$$\left|\frac{1}{R(z)}\right| \leq e^{\frac{2r}{1-r}}.$$

Hence, $|R(z)| \geq e^{\frac{-2r}{(1-r)}}$. The case of equality is attained by one of the functions of the form

$$R_\zeta(z) = \frac{1}{R_0(\zeta z)},\ |\zeta| = 1.$$

ii) and iii): Differentiation $R(z)$ in (2.5) with respect to z and $\bar{z}$ respectively yields

$$R_z(z) = R(z)\frac{1}{(1-a(z))}.\frac{p'(z)}{p(z)} \tag{2.9}$$

and

$$R_{\bar{z}}(z) = R(z)\frac{a(z)}{(1-a(z))}.\frac{p'(z)}{p(z)}. \tag{2.10}$$

(ii) and (iii) follow immediately from substituting Theorem 2.3(i), (2.7) and (2.8) in to (2.9) and (2.10). □

3 Close-to-starlike logharmonic mappings

Let $F = z|z|^{2\beta} h\overline{g}$ be logharmonic mapping. We say that F is a close-to-starlike logharmonic mapping if F is the product of a starlike logharmonic mapping $f = z|z|^{2\beta} .h^*\overline{g^*} \in S^*_{Lh}$ which is a solution of (1.1) with respect to $a \in B$ and a logharmonic mapping with positive real part $R \in P_{Lh}$ where its second dilatation function is the same a.

The geometrical interpretation is the following:under a close-to-starlike logharmonic mapping $F(z)$, the radius vector of the image of $|z| = r < 1$, never turns back by an amount more than π.

Denote by CST_{Lh} the set of all close-to-starlike logharmonic mappings. It contains in particular the set CST of all analytic close-to-starlike functions which has been introduced by Reade 1955 [6]. Also, the set S^*_{Lh} of all starlike univalent logharmonic mappings is a subset of CST_{Lh} (take $R(z) \equiv 1$ in the product). Furthermore, if $F = z|z|^{2\beta} h\overline{g}$ is a logharmonic mapping with respect to $a \in B$ satisfying $h(0) = g(0) = 1$ and $Re\, \dfrac{F(z)}{z|z|^{2\beta}} > 0$, then F is a close-to-starlike logharmonic mapping where $f(z) = z|z|^{2\beta} \left| exp\left(\int_0^z \dfrac{a(s)/s}{1-a(s)} ds\right)\right|^2$. On the other hand, a mapping $F \in CST_{Lh}$ need not to be necessarily univalent. For example, take $F(z) = z(1+z)$ where $z \in S^*$ and $1 + z \in P$.

We start this section with a representation theorem. We associate to each $F = z|z|^{2\beta} h\overline{g} \in CST_{Lh}$, the analytic function $\psi = zh/g \in CST$.

Theorem 3.1. *a) Let F be in CST_{Lh}, then $\psi \in CST$.*

b) Given any $\psi \in CST$ and $a \in B$, there are h and g in $H(U)$ uniquely determined such that

i) $0 \notin h.g(U); h(0) = g(0) = 1$

ii) $\psi = zh/g$

iii) $F = z|z|^{2\beta} h\overline{g}$ is in CST_{Lh} which is a solution of (1.1) with respect to the given a.

Proof: a) Let $F = z|z|^{2\beta} h\overline{g}$ be in CST_{Lh}. Then there exists $f = z|z|^{2\beta} h^*\overline{g^*} \in S^*_{Lh}$ and $R(z) = H\overline{G} \in$ such that

$$F(z) = f(z)R(z) = z|z|^{2\beta} h^*.\overline{g^*}.H.\overline{G}.$$

We deduce from Theorem A that $\phi = \dfrac{zh^*}{g^*} \in S^*$ and from Theorem 2.1 that $p(z) = \dfrac{H}{G} \in P$. Therefore, $\dfrac{zh^*H}{g^*G}$ is an close-to-starlike analytic map.

b) Let ψ be in CST and let $a \in B$ be given. Define

$$g(z) = exp \int_0^z \frac{sa(s)\psi'(s) + a(s).\beta.\psi(s) - \overline{\beta}\psi(s)}{s\psi(s)(1-a(s))} ds, \tag{3.1}$$

and

$$h(z) = \psi(z)g(z)/z$$

$$F = z|z|^{2\beta} h(z)\overline{g(z)} = \psi(z)|z|^{2\beta}|g(z)|^2. \tag{3.2}$$

Then h and g are nonvanishing analytic functions defined on U, normalized by $h(0) = g(0) = 1$ and f is a solution of (1.1) with respect to the given a. It is left to show that $f \in CST_{Lh}$. Since $\psi \in CST$, there exists $\phi \in S^*$ and $p \in P$ such that

$$\psi = \phi p. \tag{3.3}$$

Substituting (3.3) in (3.1) and then in (3.2) we obtain

$$F(z) = \phi(z)|z|^{2\beta}|g^*(z)|^2 p(z)|G(z)|^2,$$

where

$$g^*(z) = exp\left(\int_0^z \frac{sa(s)\phi'(s) + a(s).\beta.\phi(s) - \overline{\beta}\phi(s)}{s.\phi(s).(1-a(s))}ds\right),$$

and

$$G(z) = exp\left(\int_0^z \frac{a(s)}{1-a(s)}\frac{p'(s)}{p(s)}ds\right).$$

From Theorem A, it follows that

$$f(z) = \phi(z)|z|^{2\beta}|g^*(z)|^2 \in S^*_{Lh}$$

and from Theorem 2.1,

$$R(z) = p(z)|G(z)|^2 \in P_{Lh}.$$

This implies that $F(z) = f(z)R(z) \in CST_{Lh}$. □

It is well known that $f \in S^*$ if and only if $f(rz)/r \in S^*$ for all $r \in (0,1)$ and that the same property holds for the class P. Therefore, we have $\psi \in CST$ if and only if $\psi(rz)/r \in CST$ for all $r \in (0,1)$. Applying Theorem 3.1 we get immediately

Corollary 3.2. *$F \in CST_{Lh}$ if and only if $F(rz)/r \in CST_{Lh}$ for all $r \in (0,1)$.*

In [2] it was shown that mappings belong to S^*_{Lh} if and only if there are probability measures λ and ν on the Borel σ-algebra of ∂U and there is an $a(0) \in U$ such that

$$f(z) = z|z|^{2\beta}exp\int\int_{\partial U\times\partial U} K_2(z,\eta,\zeta;a(0))d\nu(\eta)d\lambda(\zeta), \tag{3.4}$$

where

$\beta = \overline{a(0)}(1+a(0))/(1-|a(0)|^2)$,

$$K_2(z,\eta,\zeta;a(0)) = -2log(1-\eta z) + 2Re\; log(1-\eta z) + T(z,\eta,\zeta;a(0));$$

$$T(z,\eta,\zeta;a(0)) = 2Re\{\frac{(1+a(0))(1-\overline{a(0)})\eta + (1+\overline{a(0)})(1-a(0))\zeta}{(\eta-\zeta)|1-a(0)|^2} log\frac{(1-\overline{\zeta}z)}{(1-\eta z)};$$

$if\ |\eta| = |\zeta| = 1,\ \eta \neq \zeta\}$

and

$$T(z,\eta,\eta;a(0)) = 4Re\left(\frac{\eta z}{1-\eta z}\frac{1-|a(0)|^2}{|1-a(0)|^2}\right).$$

Together with Theorem 2.2 one can characterize mappings in CST_{Lh} by an appropriate integral representation.

In the next two results we determine the radius of univalence and the radius of starlikeness for the mappings in the set CST_{Lh} and for the mappings in the logarithmic convex combination of the sets CST_{Lh} and S^*_{Lh}.

Theorem 3.3. *Let $F = z|z|^{2\beta}h\overline{g} \in CST_{Lh}$. Then F maps the disk $|z| < R$, $R \leq 2-\sqrt{3}$ onto a starlike domain. The upper bound is best possible for all $a \in B$.*

Proof: Let $F = z|z|^{2\beta}h\overline{g} \in CST_{Lh}$ with respect to a given $a \in B$. Then there exists a function $f = z|z|^{2\beta}h^*\overline{g^*} \in S^*_{Lh}$ and a function $R(z) = H.\overline{G} \in P_{Lh}$ such that both functions are logharmonic with respect to the same a and that

$$F(z) = f(z)R(z). \tag{3.5}$$

Now, Theorem A implies that $\phi(z) = \frac{zh^*}{g^*} \in S^*$ and then

$$f(z) = \phi(z)|z|^{2\beta} exp\, 2Re \int_0^z \frac{a(s)}{1-a(s)} \frac{\phi'(s)}{\phi(s)} ds. \tag{3.6}$$

Also, it follows from Theorem 2.1 that $p(z) = H(z)/G(z) \in P$ and then

$$R(z) = p(z) exp\, 2Re \int_0^z \frac{a(s)}{1-a(s)} \frac{p'(s)}{p(s)} ds. \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.5), then simple calculations lead to

$$Re\frac{zF_z - \overline{z}F_{\overline{z}}}{F} = Re\frac{zf_z - \overline{z}f_{\overline{z}}}{f} + Re\frac{zR_z - \overline{z}R_{\overline{z}}}{R} \tag{3.8}$$

$$= Re\frac{z\phi'}{\phi} + Re\frac{zp'}{p}.$$

Since

$$Re\frac{z\phi'}{\phi} \geq \frac{1-|z|}{1+|z|} \text{ and } Re\frac{zp'}{p} \geq \frac{-2|z|}{1-|z|^2},$$

we have

$$Re\frac{zF_z - \overline{z}F_{\overline{z}}}{F} \geq \frac{1-|z|}{1+|z|} - \frac{2|z|}{1-|z|^2} = \frac{|z|^2 - 4|z| + 1}{1-|z|^2}. \tag{3.9}$$

Thus, $Re\frac{zF_z - \overline{z}F_{\overline{z}}}{F} > 0$ if $1 - 4|z| + |z|^2 > 0$. The radius of starlikeness ρ is the smallest positive root (less than 1) of $\rho^2 - 4\rho + 1 = 0$ which is $2 - \sqrt{3}$. Therefore, F is univalent on $|z| < 2 - \sqrt{3}$ and maps $\{z; |z| < 2 - \sqrt{3}\}$ onto a starlike domain, The analytic function $f(z) = \frac{z(1+z)}{(1-z)^2}$ belongs to the set CST and hence to the set CST_{Lh} and we have $f'(\sqrt{3} - 2) = 0$. Hence, the upper bound $2 - \sqrt{3}$ is best possible for CST. Since $f = z|z|^{2\beta}h^*\overline{g^*} \in S^*_{Lh}$ if and only if $zh^*/g^* \in S^*$ (Theorem A) the same bound is best possible for all $a \in B$. □

Remark. The minimum of the first term on the right hand side of equation (3.8) is attained for the function $f(z) = \overline{\zeta} f_0(\zeta z)$, $|\zeta| = 1$, where

$$f_0(z) = \frac{z(1+\overline{z})}{(1+z)} exp\left(Re\frac{-4z}{1-z}\right)$$

plays the rule of the Koebe mapping in the set of univalent logharmonic mappings. Indeed, by simple calculations we obtain that $Re\frac{zf_z - \overline{z}f_{\overline{z}}}{f} = Re\frac{1-\zeta z}{1+\zeta z}$.

Theorem 3.4. *Let $F = z|z|^{2\beta}h\overline{g} \in CST_{Lh}$ with respect to a given $a \in B$ and let $f = z|z|^{2\beta}h^*\overline{g^*} \in S^*_{Lh}$ with respect to the same a. Then $S(z) = f(z)^\lambda F(z)^{1-\lambda}, 0 < \lambda < 1$ is univalent and starlike in $|z| < 2 -$ The bound is best possible for all $a \in B$.*

Proof: Let $S(z) = f(z)^{\lambda}F(z)^{1-\lambda}$, $0 < \lambda < 1$ where $f = z|z|^{2\beta}h^*\overline{g^*} \in S^*_{Lh}$ and $F = z|z|^{2\beta}h\overline{g} \in CST_{Lh}$. Both mappings are logharmonic with respect to the same a. Then $S(z)$ is a logharmonic mapping with respect to the same a. Moreover, we have

$$Re\frac{zS_z - \overline{z}S_{\overline{z}}}{S} = \lambda Re\frac{zf_z - \overline{z}f_{\overline{z}}}{f} + (1-\lambda)Re\frac{zF_z - \overline{z}F_{\overline{z}}}{F}. \tag{3.10}$$

Substituting from (3.6) and (3.9) into (3.10), we deduce

$$Re\frac{zS_z - \overline{z}S_{\overline{z}}}{S} \geq \lambda\left(\frac{1-|z|}{1+|z|}\right) + (1-\lambda)\left(\frac{|z|^2 - 4|z| + 1}{1-|z|^2}\right)$$
$$= \frac{|z|^2 + (2\lambda - 4)|z| + 1}{1-|z|^2}.$$

Thus, $Re\frac{zS_z - \overline{z}S_{\overline{z}}}{S} > 0$ if $|z|^2 + (2\lambda-4)|z| + 1 > 0$. The last inequality is satisfied for $|z| < 2 - \lambda - \sqrt{}$ Therefore, $S(z)$ is univalent in $|z| < 2 - \lambda - \sqrt{\lambda^2 - 4\lambda + 3}$ and maps that circle onto a starlike domain. The function

$$S(z) = f_0(z)^{\lambda}F_0(z)^{1-\lambda},$$

where

$$f_0(z) = \frac{z}{(1+z)^2}$$

and

$$F_0(z) = \frac{z(1-z)}{(1+z)^3}$$

satisfies the hypothesis of the theorem because $f_0(z)$ belongs to the set S^* and therefore, to the set S^*_{Lh} and also since $F_0(z)$ belongs to the set CST and hence to the set CST_{Lh}. But for this function $S'(2 - \lambda - \sqrt{\lambda^2 - 4\lambda + 3}) = 0$. Therefore, the upper bound $2 - \lambda - \sqrt{\lambda^2 - 4\lambda + 3}$ is best possible for the set $\{S(z)|S(z) = f(z)^{\lambda}F(z)^{1-\lambda};\ f \in S^*_{Lh}\ and\ F \in CST_{Lh}\}$. From Theorem A and Theorem 3.1, we deduce the same bound is best possible for all $a \in B$. □

Our next result is a distortion theorem for the subset S^*_{Lh} for which $\beta = 0$, i.e. $a(0) = 0$.

Theorem 3.5. *Let $f = zh^*\overline{g^*} \in S^*_{Lh}$. Then for every $z \in U$ we have*

i) $|z|exp\left(\frac{-4|z|}{1+|z|}\right) \leq |f(z)| \leq |z|exp\left(\frac{4|z|}{1-|z|}\right)$

ii) $\frac{(1-|z|)}{(1+|z|)^2}exp\left(\frac{-4|z|}{1+|z|}\right) \leq |f_z(z)| \leq \frac{(1+|z|)}{(1-|z|)^2}exp\left(\frac{4|z|}{1-|z|}\right)$

iii) $|f_{\overline{z}}(z)| \leq \frac{|z|(1+|z|)}{(1-|z|)^2}exp\left(\frac{4|z|}{1-|z|}\right)$.

The equalities hold if $f(z)$ is one of the functions of the form $\overline{\zeta}f_0(\zeta z)$, $|\zeta| = 1$, where

$$f_0(z) = \frac{z(1-\overline{z})}{(1-z)}exp\left(Re\frac{4z}{1-z}\right).$$

Proof:i) Let $f = zh^*\overline{g^*} \in S^*_{Lh}$. Then it follows from Theorem A that f admits the representation

$$f(z) = \phi(z)exp\left(2Re\int_0^z \frac{a(s)}{(1-a(s))}\frac{\phi'(s)}{\phi(s)}ds\right), \tag{3.11}$$

where $a \in B$ with $a(0) = 0$. For $|z| = r$ we have

$$|z\phi'(z)/\phi(z)| \leq (1+r)/(1-r), \tag{3.12}$$

$$|a(z)/[z(1-a(z))]| \leq 1/(1-r), \tag{3.13}$$

and

$$|\phi(z)| \leq r/(1-r)^2.$$

Therefore,

$$|f(z)| \leq \frac{r}{(1-r)^2} exp\left(2\int_0^r (1+t)/(1-t)^2 dt\right) = rexp\left(\frac{4r}{1-r}\right).$$

Equality occurs if and only if $a(z) = \zeta z$ and $\phi(z)/(1-\zeta z)^2$, $|\zeta| = 1$, which leads to $f(z) = \overline{\zeta} f_0(\zeta z)$.

For the left hand side inequality, consider the integral representation (3.4) with $\beta = 0$($resp.$ $a(0) = 0$). Then

$$f(z) = zexp \int_{\partial U \times \partial U} K_2(z, \eta, \zeta; 0) d\nu(\eta) d\lambda(\zeta)$$

where

$$K_2(z,\eta,\zeta;0) = \begin{cases} log\frac{(1-\overline{\eta z})}{(1-\eta z)} - 2Im\left[\frac{\eta+\zeta}{\eta-\zeta}\right] arg\frac{1-\zeta z}{1-\eta z}; & |\eta| = |\zeta| = 1 \text{ } and \text{ } \eta \neq \zeta \\ log\frac{(1-\overline{\eta z})}{(1-\eta z)} + 4Re\left[\frac{\eta z}{1-\eta z}\right]. & \end{cases}$$

For $|z| = r$ we have

$$\begin{aligned} log|f(z)/z| &= \{Re \int_{\partial U \times \partial U} K_2(z,\zeta,\eta;0) d\nu(\eta)\lambda(\zeta)\} \\ &\geq \min_{\nu,\lambda}\{\min_{|z|=r} Re \int_{\partial U \times \partial U} K_2(z,\zeta,\eta;0) d\nu(\eta)\lambda(\zeta)\} \\ &= min\{\min_{0<|\ell|\leq\pi/2} -2Im\frac{(1+e^{2i\ell})}{(1-e^{2i\ell})} arg[\frac{1-e^{2i\ell}z}{1-z}]; -4r/(1+r)\}, \end{aligned}$$

where $e^{2i\ell} = \overline{\eta}\zeta$. Put

$$\Phi_r(\ell) = \begin{cases} \min_{|z|=r} -2Im\left(\frac{1+e^{2i\ell}}{1-e^{2i\ell}}\right) arg\left(\frac{1-e^{2i\ell}z}{1-z}\right) & \text{if } 0 < |\ell| \leq \pi/2 \text{ } and \\ -4r/(1+r) & \text{if } \ell = 0. \end{cases}$$

Then $\Phi_r(\ell)$ is a continuous and even function on $|\ell| < \pi/2$. Hence

$$log\left|\frac{f(z)}{z}\right| \geq \min_{0\leq\ell\leq\pi/2} \Phi_r(\ell) = \inf_{0<\ell<\pi/2} \Phi_r(\ell).$$

Since

$$\max_{|z|=r} arg\left(\frac{1-e^{2i\ell}z}{1-z}\right) = 2arctan\left(\frac{rsin\ \ell}{1+rcos\ \ell}\right),$$

we get

$$log\left|\frac{f(z)}{z}\right| \geq \inf_{0<\ell<\pi/2} -4cot\ \ell.arctan\left(\frac{rsin\ \ell}{1+rcos\ \ell}\right)$$

and using the fact that $|arctan\ x| \leq |x|$, we have

$$log\left|\frac{f(z)}{z}\right| \geq \inf_{0<\ell<\pi/2}\left(\frac{-4r\cos\ell}{1+r\cos\ell}\right) \geq \frac{-4r}{(1+r)}.$$

The case of equality is attained by one of the functions of the form $\overline{\zeta}f_0(\zeta z)$; $|\zeta|=1$.

ii) and iii) Differentiation $f(z)$ in (3.11) with respect to z and $\overline{z}$ respectively leads to

$$f_z(z) = f(z)\frac{1}{1-a(z)}\frac{\phi'(z)}{\phi(z)} \tag{3.14}$$

and

$$f_{\overline{z}}(z) = f(z)\frac{a(z)}{1-a(z)}\frac{\phi'(z)}{\phi(z)}. \tag{3.15}$$

The result follows from substituting from Theorem 3.5(i),(3.12) and (3.13) in to (3.14) and (3.15). □

Combining Theorem 2.3 and Theorem 3.5 together with (3.5) we deduce the following distortion theorem for the set CST_{Lh}.

Theorem 3.6. *Let $F = zh\overline{g} \in CST_{Lh}$. Then for every $z \in U$ we have*

i) $|z|exp\left[\frac{-2|z|}{1-|z|} - \frac{4|z|}{1+|z|}\right] \leq |F(z)| \leq |z|exp\left(\frac{6|z|}{1-|z|}\right)$

ii) $|F_z(z)| \leq \frac{|z|^2+4|z|+1}{(1-|z|)^2(1+|z|)}exp\left(\frac{6|z|}{1-|z|}\right)$

iii) $|F_{\overline{z}}(z)| \leq \frac{|z|(|z|^2+4|z|+1)}{(1-|z|)^2(1+|z|)}exp\left(\frac{6|z|}{1-|z|}\right).$

Equality holds for the right hand side inequalities if $F(z)$ is one of the functions of the form

$$F_{\eta,\zeta}(z) = \frac{z(1-\overline{\eta}\overline{z})}{(1-\eta z)}\frac{(1+\zeta z)}{(1-\zeta z)}\left|\frac{1-\zeta z}{1+\zeta z}\right| exp\left(Re\left[\frac{4\eta z}{1-\eta z}+\frac{2\zeta z}{1-\zeta z}\right]\right)$$

where $|\eta| = |\zeta| = 1$, and for the left hand side inequality if $F(z)$ is one of the functions of the form

$$F_{\eta,\zeta}(z) = \frac{z(1-\overline{\eta}\overline{z})}{(1-\eta z)}\frac{(1+\zeta z)}{(1-\zeta z)}\left|\frac{1-\zeta z}{1+\zeta z}\right| exp\left(Re\left[\frac{4\eta z}{1-\eta z}-\frac{2\zeta z}{1-\zeta z}\right]\right),$$

where $|\eta| = |\zeta| = 1$.

Finally, we prove the following theorem.

Theorem 3.7. *Let $F = z|z|^{2\beta}h\overline{g} \in CST_{Lh}$. Then we have*

$$\left|arg\frac{F(z)}{z}\right| \leq 2arcsin\ |z| + arcsin\ \frac{|z|}{(1+|z|^2)} + 2Im(\beta)\ ln|z|.$$

Equality holds if and only if

$$\psi(z) = \frac{zh}{g} = \frac{z(1+\eta z)}{(1-\eta z)^2},\ |\eta| = 1$$

and

$$p(z) = \frac{1+\zeta z}{1-\zeta z},\ |\zeta| = 1.$$

Proof: Let $F = z|z|^{2\beta}h\overline{g} \in CST_{Lh}$. Then $\psi = zh/g \in CST$ by Theorem 3.1. But since $\psi(z) = \phi(z)p(z)$, where $\phi \in S^*$ and $p \in P$. The result follows immediately from

$$arg\frac{F(z)}{z} = arg\frac{\psi(z)}{z} + 2Im\ \beta ln|z| = arg\frac{\phi(z)}{z} + arg\ p(z) + 2Im\ \beta ln|z|$$

and from [3, p.71]. □

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References

[1] Abdulhadi, Z. and Bshouty, D. Univalent Mappings in $H.\overline{H}$, Trans. of Amer. Math. Soc. 305, (1988), 841-849.

[2] Abdulhadi, Z. and Hengartner, W. Spirallike logharmonic mappings, Complex Variables Theory Applications, 9, (1987), 121-130.

[3] Duren, P. L. Univalent Functions, NewYork, Feb. 1983.

[4] Nitsche, J.C.C. Lectures on minimal surfaces, Vol. 1, NewYork, 1989.

[5] Osserman, R. A survey of Minimal Surfaces, Dover, 1986.

[6] Reade, M. On close to convex univalent functions, Michigan Math. J., 3, (1955), 59-62.