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SOME COUNTEREXAMPLES AND PROPERTIES ON GENERALIZATIONS OF LINDELÖF SPACES

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ABSTRACT. In this paper we give some significative counterexamples to prove that some well known generalizations of Lindelof property are proper Also we give some new results, we introduce and study the almost regular-Lindelof spaces as a generalization of the almost-Lindelöf spaces and as a new and significative application of the quasi-regular open subsets of [1].

KEY WORDS AND PHRASES: Lindelof space, almost Lindelof, weakly Lindelof and nearly Lindelof, semiregular and almost-regular space, regular cover
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1. INTRODUCTION

In literature there are several generalizations of the notion of Lindelof space [2] and these are studied separately for different reasons and purposes In 1959 Frolik [3] introduced the notion of weakly-Lindelöf spaces that, afterward, was studied by several authors: Comfort, Hindman and Negrepointis [4] in 1969, Hager [5] in 1969, Ulmer [6] in 1972, Woods [7] in 1976, Bell, Ginsburg and Woods [8] in 1978 About this topic in 1982 Balasubramanian [9] introduced and studied the notion of nearly-Lindelof spaces that is between Lindelöf and weakly-Lindelof spaces In 1984 Willard and Dissanayake [10] gave the notion of almost k-Lindelöf space, that for $k = \aleph_0$ we call almost-Lindelof, and that is between the nearly-Lindelof and weakly-Lindelof spaces. To be complete, it is useful to recall some recent papers of Pareek [11] which are an almost survey of all main generalizations of Lindelof spaces

In this paper we fix our attention on the main generalizations of Lindelöf spaces, i e weakly-Lindelof, almost-Lindelöf and nearly-Lindelöf spaces Our purpose is to study the relations between them and some new properties but, mainly, to construct some significative counterexamples to prove that the studied generalizations are proper.

Moreover, the counterexample 3.11, proving that there exist weakly-Lindelof spaces not almost-Lindelof, guides us to introduce and study a new generalization of Lindelof spaces, i e the almost regular-Lindelöf spaces These almost regular-Lindelöf spaces are a new and significative application of quasi-regular open subset introduced by the first author and Lo Faro [1] in 1981

We conclude the paper proposing the study of two new and natural generalizations of the almost regular-Lindelöf spaces, i e the weakly regular-Lindelof and the nearly regular-Lindelof spaces

In particular, this paper is composed of four parts In §1 we study the nearly-Lindelof spaces as a generalization of Lindelof spaces (while Balasubramanian has studied them as a generalization of nearly compact spaces), we give some properties and a counterexample of a nearly-Lindelöf not Lindelof space In §2 we study the subspaces and subsets nearly-Lindelof relative In §3 we give some properties of weakly-Lindelof spaces and a counterexample of weakly-Lindelof not nearly-Lindelof space, moreover, we study the almost-Lindelof spaces that are between nearly-Lindelof and weakly-Lindelof spaces, we give the necessary counterexamples and properties In the last section we introduce the notions of almost regular-Lindelof, weakly regular-Lindelof and nearly regular-Lindelof spaces

We have that the following implications hold·

$$\begin{array}{ccccc} \text{Nearly-L} & \Rightarrow & \text{Almost-L} & \Rightarrow & \text{Weakly-L} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{Nearly R -L} & \Rightarrow & \text{Almost R -L} & \Rightarrow & \text{Weakly R -L} \end{array}$$

PRELIMINARIES

Throughout the present paper X and Y always denote topological spaces, x an element of X and $\mathcal{U}_x$ the neighborhoods filter of x in X The interior and the closure of any subset A of X will be denoted by int(A) or $\mathring{A}$ and cl(A) or $\overline{A}$ respectively.

If $A \subseteq S \subseteq X$, then $\text{int}_S(A)$ and $\text{cl}_S(A)$ will be used to denote respectively the interior and closure of A in the subspace S With $\{a_i\}_{i\geq\alpha}$ and $\{a_i\}_{i\in\mathbb{N}}$ we denote the set of all elements a_i for each $i \geq \alpha$ and for each $i \in \mathbb{N}$ respectively

Recall some definitions

DEFINITION 1. A subset $A \subseteq X$ is called *regularly open* (resp. *regularly closed*) if $A = \mathring{\overline{A}}$ (resp $A = \overline{\mathring{A}}$)

The topology generated by the regularly open subsets of the space (X, τ) is denoted by τ^* and it is called *semiregularization* of X, if $\tau \equiv \tau^*$ then X is said to be *semiregular* [12].

DEFINITION 2 [13]. A topological space X is said to be *almost regular* if for each $x \in X$ and each regularly open neighborhood $U_x \in \mathcal{U}_x$, there exists a neighborhood $V_x \in \mathcal{U}_x$ such that $V_x \subset \overline{V}_x \subset U_x$, or, equivalently, if for any regularly closed set C and any singleton $\{x\}$ disjoint from C, there exist disjoint open sets U and V such that $C \subseteq U$ and $x \in V$.

It is true that a space X is regular if and only if it is semiregular and almost regular [13]

DEFINITION 3 [14]. A topological space X is said to be *nearly compact* if every open cover $\{U_\lambda\}_{\lambda\in\Lambda}$ of X admits a finite subfamily such that $X = \bigcup_{i=1}^{n} \mathring{\overline{U}}_{\lambda_i}$.

DEFINITION 4 [2] Let X be a topological space. A cover $\mathcal{V} = \{V_j\}_{j\in J}$ of X is a *refinement* of another cover $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ if for each $j \in J$ there exists an $\lambda(j) \in \Lambda$ such that $V_j \subset U_{\lambda(j)}$.

DEFINITION 5 [2]. A family $\{U_\lambda\}_{\lambda\in\Lambda}$ of subsets of a topological space X is *locally finite* if for every point $x \in X$ there exists a neighborhood $U_x \in \mathcal{U}_x$ such that the set $\{\lambda \in \Lambda : U_x \cap U_\lambda \neq \emptyset\}$ is finite

§1. NEARLY LINDELÖF-SPACES

DEFINITION 1.1 [9]. A topological space X is said to be *nearly-Lindelöf* if for every open cover $\{U_\lambda\}_{\lambda\in\Lambda}$ of X there exists a countable subset $\{\lambda_n\}_{n\in\mathbb{N}}$ of Λ such that $X = \bigcup_{n\in\mathbb{N}} \mathring{\overline{U}}_{\lambda_n}$ (i e if every cover of X by regularly open sets admits a countable subcover).

It is clear that every compact space is nearly-Lindelöf, but the converse is not true (for example the real line $\mathbb{R}$ is nearly-Lindelöf but it is not nearly compact).

Moreover, every Lindelöf space is nearly-Lindelöf but the converse is not true as the following example shows.

EXAMPLE 1.2. Let Ω be the smallest uncountable ordinal number and $A = [0, \Omega)$. The set A has the property that for each $\alpha \in A$ the set $[0, \alpha)$ is countable, while A is not. Let $X = \{a_{ij}, c_i, a\}$ where $i \in A$ and $j \in \mathbb{N}$. We define in X a topology such that the points $\{a_{ij}\}$ are isolated and the fundamental system of neighborhoods of the points $\{c_i\}$ and $\{a\}$ are

$$B^n_{c_i} = \{c_i, a_{ij}\}_{j\geq n} \quad \text{and} \quad B^\alpha_a = \{a, a_{ij}\}_{i\geq\alpha, j\in\mathbb{N}}$$

respectively X so topologized is Hausdorff but not Lindelof, in fact the open cover $\{B_a^1\} \cup \{B_{c_i}^0\}_{i \in A}$ has not countable subcover On the other hand, X is nearly-Lindelof In fact, let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a cover of X and $\bar{\lambda}$ such that $a \in U_{\bar{\lambda}}$ Then $\left(X \backslash \overset{o}{\overline{U_{\bar{\lambda}}}}\right)$ is a countable set It follows that X is nearly-Lindelof □

PROPERTY 1.3. A space (X, τ) is nearly-Lindelof if and only if (X, τ^*) is Lindelof □

COROLLARY 1.4. A nearly-Lindelof space (X, τ) is Lindelof if and only if it is semiregular □

This is an improvement of [prop 5, g] that holds only for regular spaces

PROPOSITION 1.5 [9] A topological space X is nearly-Lindelof if and only if for any family $\{C_\lambda\}_{\lambda \in \Lambda}$ by regularly closed sets of X with countable intersection property, the intersection $\bigcap_{\lambda \in \Lambda} C_\lambda$ is non-empty □

PROPOSITION 1.6. Let X be an almost regular and nearly-Lindelof space Then for every disjoint regularly closed C_1 and C_2 there exist two open sets U and V such that $U \cap V = \emptyset$ and $C_1 \subset U, C_2 \subset V$

PROOF. Since X is almost regular, for each $x \in C_1$ there exists an open neighborhood U_x such that $\overline{U_x} \cap C_2 = \emptyset$ We can suppose U_x to be regularly open The family $\{U_x\}_{x \in C_1} \cup \{X \backslash C_1\}$ is a regularly open cover of X and, since X is nearly-Lindelof, there exists a countable set of points $x_1, x_2, \dots, x_n, \dots$ of X such that $X = \left(\bigcup_{n \in \mathbb{N}} U_{x_n}\right) \cup (X \backslash C_1)$ It follows that for each $n \in \mathbb{N}$ $C_1 \subset \bigcup_{n \in \mathbb{N}} U_{x_n}$ and $\overline{U_{x_n}} \cap C_2 = \emptyset$ Analogously there exists a family of regular open sets $\{V_{y_n}\}$ such that $C_2 \subset \bigcup_{n \in \mathbb{N}} V_{y_n}$ and $\overline{V_{y_n}} \cap C_1 = \emptyset$ Let $G_n = U_{x_n} \backslash \left(\bigcup_{i=1}^{n} \overline{V_{y_i}}\right), H_n = V_{y_n} \backslash \left(\bigcup_{i=1}^{n} \overline{U_{x_i}}\right)$ and $U = \bigcup_{n \in \mathbb{N}} G_n$, $V = \bigcup_{n \in \mathbb{N}} H_n$ U and V so constructed are the open sets that we are looking for □

DEFINITION 1.7 [15] A space X is said to be *nearly paracompact* if every cover of X by regularly open sets admits a locally finite refinement

PROPOSITION 1.8 Let X be an almost regular and nearly-Lindelof space Then X is nearly paracompact

PROOF Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a cover of X by regularly open sets For each $x \in X$ and $\bar{\lambda} \in \Lambda$ such that $x \in U_{\bar{\lambda}}$ there exists an open neighborhood U_x of x such that $\overline{U_x} \subset U_{\bar{\lambda}}$ We can suppose that U_x is regularly open so $\{U_x\}_{x \in X}$ is a regular open cover of X. Since X is nearly-Lindelöf, there exists a countable set of points $x_1, x_2, \dots, x_n, \dots$ of X such that $X = \bigcup_{n \in \mathbb{N}} U_{x_n}$ For each $n \in \mathbb{N}$ choose a $\lambda_n \in \Lambda$ such that $\overline{U_{x_n}} \subset U_{\lambda_n}$ and put $V_n = U_{\lambda_n} \backslash \left(\bigcup_{i=1}^{n-1} \overline{U_{x_i}}\right)$. By construction $\{V_n\}_{n \in \mathbb{N}}$ is a refinement of $\{U_\lambda\}_{\lambda \in \Lambda}$ and it is a locally finite family. In fact, let $x \in X$. Then there exist U_{x_p} (since $\{U_{x_n}\}_{n \in \mathbb{N}}$ is a cover of X) and U_{λ_p} such that $x \in U_{x_p} \subset U_{\lambda_p}$. We will prove that U_{x_p} intersects at most finitely many members of the family $\{V_n\}_{n \in \mathbb{N}}$ Since

$$V_1 = U_{\lambda_1}, V_2 = U_{\lambda_2} \backslash \overline{U_{x_1}}, \dots, V_{p+1} = U_{\lambda_{p+1}} \backslash \{\overline{U_{x_1}} \cup \dots \cup \overline{U_{x_p}}\},$$

then U_{x_p} is not contained in V_r for each $r \geq p+1$ while $U_{x_p} \subset V_p$. So $U_{x_p} \cap V_r = \emptyset$ for each $r \geq p+1$, therefore U_{x_p} intersects at most a finite number of sets V_n. □

PROPOSITION 1.9. Let X be a nearly-Lindelöf space and Y a nearly compact space Then $X \times Y$ is nearly-Lindelof.

PROOF. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a cover of $X \times Y$ by regularly open sets Without loss of generality, we can suppose $U_\lambda = V_\lambda \times W_\lambda$ where V_λ and W_λ are regularly open sets in X and Y respectively Fix $x \in X$, for each $y \in Y$ there exists $\lambda_y \in \Lambda$ such that $(x, y) \in V_{\lambda_y} \times W_{\lambda_y}$ The family $\{W_{\lambda_y}\}_{y \in Y}$ is a cover of Y by regularly open sets and, since Y is nearly compact, it admits a finite subcover Let $Y = W_{\lambda_{y_1}} \cup \dots \cup W_{\lambda_{y_n}}$ Put $H_x = V_{\lambda_{y_1}} \cap \dots \cap V_{\lambda_{y_r}}$ and $r(x) = \{\lambda_{y_1}, \dots, \lambda_{y_r}\}$ H_x is a regularly open set of X and hence the family $\{H_x\}_{x \in X}$ is a regularly open cover of X Since X is nearly-Lindelof, there exists a countable set of points $x_1, x_2, \dots, x_n, \dots$ of X such that $X = \bigcup_{n \in \mathbb{N}} H_{x_n}$, hence

$$X \times Y = \left(\underset{n \in N}{\cup} H_{x_n} \right) \times Y = \underset{n \in N, i \in r(x_n)}{\cup} (H_{x_n} \times W_i) = \underset{n \in N, i \in r(x_n)}{\cup} (V_i \times W_i).$$

Since the last member is a countable family, we have that $X \times Y$ is nearly-Lindelof □

REMARK 1.10 In general the product of two nearly-Lindelof spaces is not nearly-Lindelof In fact, let K be the Sorgenfrey line K is normal, and hence regular, and Lindelof and therefore it is nearly-Lindelof The product $K \times K$ is regular, but it is not Lindelöf [2, 3 8 15] and therefore it cannot be nearly-Lindelof (see Corollary 1 4)

§2. NEARLY-LINDELÖF SUBSPACES AND SUBSETS

A subset S of a space X is said to be nearly-Lindelof if S is nearly-Lindelof as subspace of X (i e S is nearly-Lindelöf with respect to the inducted topology from the topology of X)

DEFINITION 2.1 A subset S of a space X is said to be *nearly-Lindelöf relative to X* if for every cover $\{U_\lambda\}_{\lambda \in \Lambda}$ by open sets of X such that $S \subset \underset{\lambda \in \Lambda}{\cup} U_\lambda$, there exists a countable subset $\{\lambda_n\}_{n \in N} \subset \Lambda$ such that $S \subset \underset{n \in N}{\cup} \overset{\circ}{\overline{U_{\lambda_n}}}$

PROPOSITION 2.2 [9] Let X be a space and A an open subset of X Then A is nearly-Lindelof if and only if it is nearly-Lindelof relative to X □

LEMMA 2.3 [9] Let B be a regularly closed subset of a nearly-Lindelöf space X Then C is nearly-Lindelöf relative to X □

COROLLARY 2.4 [9] A clopen of a nearly-Lindelof space is nearly-Lindelof □

PROPERTY 2.5 Let X be an extremally disconnected space (i e the closure of an open set is open [2]) and $S \subseteq X$ If S is nearly-Lindelöf then S is nearly-Lindelöf relative to X.

PROOF. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open family of X such that $S \subset \underset{\lambda \in \Lambda}{\cup} U_\lambda$ Consider $V_\lambda = S \cap U_\lambda$ for each $\lambda \in \Lambda$, then $\{V_\lambda\}_{\lambda \in \Lambda}$ is an open cover of S. By hypothesis there exists a countable subfamily $\{V_{\lambda_n}\}_{n \in \mathbf{N}}$ such that $S = \underset{n \in \mathbf{N}}{\cup} \mathrm{int}_S \mathrm{cl}_S(V_{\lambda_n})$. Since for each $n \in \mathbb{N}$ $V_{\lambda_n} \subset U_{\lambda_n}$, then $\overline{V_{\lambda_n}}^S \subset \overline{U_{\lambda_n}}^X$ Since X is extremally disconnected then $\mathrm{int}_S \mathrm{cl}_S(V_{\lambda_n}) \subset \mathrm{int}_X \mathrm{cl}_X(U_{\lambda_n}) = \mathrm{cl}_X(U_{\lambda_n})$ This proves that $S \subseteq \underset{n \in \mathbf{N}}{\cup} \overset{\circ}{\overline{U_{\lambda_n}}}$, i.e. S is nearly-Lindelöf relative to X. □

REMARK 2.6. In general a closed subset of a nearly-Lindelöf space is neither nearly-Lindelöf nor nearly-Lindelöf relative to the space as the subset $\{c_i\}_{i \in A}$ in Example 1.2 shows

PROPOSITION 2.7. Let X be a space and $S \subset X$. The following are equivalent

(i) S is nearly-Lindelöf relative to X;

(ii) for every family by regularly open sets of X that cover S, there exists a countable subfamily covering S;

(iii) for every family $\{C_\lambda\}_{\lambda \in \Lambda}$ by regularly closed sets of X such that $\left(\underset{\lambda \in \Lambda}{\cap} C_\lambda \right) \cap S = \emptyset$, there exists a countable subset of indices $\{\lambda_n\}_{n \in \mathbf{N}} \subset \Lambda$ such that $\left(\underset{n \in \mathbf{N}}{\cap} C_{\lambda_n} \right) \cap S = \emptyset$

PROOF. (i) ⇒ (ii) It is obvious by the definition.

(ii) ⇒ (iii). Let $\{C_\lambda\}_{\lambda \in \Lambda}$ be a regularly closed family in X such that $\left(\underset{\lambda \in \Lambda}{\cap} C_\lambda \right) \cap S = \emptyset$. Then $S \subset X \backslash \left(\underset{\lambda \in \Lambda}{\cap} C_\lambda \right) = \underset{\lambda \in \Lambda}{\cup} (X \backslash C_\lambda)$; hence $\{X \backslash C_\lambda\}_{\lambda \in \Lambda}$ is a regularly open family covering S, then there exists a countable subfamily $\{X \backslash C_{\lambda_n}\}_{n \in \mathbf{N}}$ such that $S \subset \underset{n \in \mathbf{N}}{\cup} (X \backslash C_{\lambda_n})$, i e $\left(\underset{n \in \mathbf{N}}{\cap} C_{\lambda_n} \right) \cap S = \emptyset$

(iii) ⇒ (i) Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a family by open subsets of X such that $S \subset \underset{\lambda \in \Lambda}{\cup} U_\lambda$ Then $S \subset B2 \underset{\lambda \in \Lambda}{\cup} U_\lambda \subset \underset{\lambda \in \Lambda}{\cup} \overset{\circ}{\overline{U_\lambda}}$, therefore $\left(X \backslash \left(\underset{\lambda \in \Lambda}{\cup} \overset{\circ}{\overline{U_\lambda}} \right) \right) \cap S = \emptyset$, i e $\underset{\lambda \in \Lambda}{\cap} \left(X \backslash \overset{\circ}{\overline{U_\lambda}} \right) \cap S = \emptyset$ By hypothesis there exists a countable subfamily $\left\{ X \backslash \overset{\circ}{\overline{U_{\lambda_n}}} \right\}_{n \in \mathbf{N}}$ such that $\underset{n \in \mathbf{N}}{\cap} \left(X \backslash \overset{\circ}{\overline{U_{\lambda_n}}} \right) \cap S = \emptyset$ and therefore $\left(X \backslash \left(\underset{n \in \mathbf{N}}{\cup} \overset{\circ}{\overline{U_{\lambda_n}}} \right) \right) \cap S = \emptyset$, i.e $S \subset \underset{n \in \mathbf{N}}{\cup} \overset{\circ}{\overline{U_{\lambda_n}}}$. This completes the proof □

PROPOSITION 2.8. A space (X, τ) is open hereditarily nearly-Lindelof if and only if any $A \in \tau^*$ is nearly-Lindelof

PROOF. Let $B \subset X$ be an open subset of X By Proposition 2 2 it is sufficient to prove that B is nearly-Lindelof relative to X Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a family by regularly open sets of X such that $B \subset \underset{\lambda\in\Lambda}{\cup} U_\lambda$ The set $A = \underset{\lambda\in\Lambda}{\cup} U_\lambda$ belongs to τ^*, so by hypothesis A is nearly-Lindelof Hence there exists a countable subfamily $\{U_{\lambda_n}\}_{n\in\mathbf{N}}$ of $\{U_\lambda\}_{\lambda\in\Lambda}$ such that $A = \underset{n\in\mathbf{N}}{\cup} U_{\lambda_n}$ and therefore $B \subset \underset{n\in\mathbf{N}}{\cup} U_{\lambda_n}$ Conversely, let X be open hereditarily nearly-Lindelof Since $\tau^* \subset \tau$, it is obvious that any $A \in \tau^*$ is nearly-Lindelof □

THEOREM 2.9. Let $f : X \to Y$ be a closed continuous function and, for each $y \in Y$, let $f^{-1}(y)$ be nearly-Lindelof relative to X If Y is nearly-Lindelof then X is nearly-Lindelof

PROOF. Let $\{C_\lambda\}_{\lambda\in\Lambda}$ be a family of regularly closed subsets of X with countable intersection property Let $M = \Lambda^{\mathbf{N}}$, i e each $\mu \in M$ is of the form $\mu = (\lambda_1, \lambda_2, ..., \lambda_n, ...)$ Put $C_\mu = \underset{n\in\mathbf{N}}{\cap} C_{\lambda_n} \neq \emptyset$ The family $\{C_\mu\}_{\mu\in M}$ is a family by closed subsets of X with countable intersection property and also the family $\{f(C_\mu)\}_{\mu\in M}$ in Y is so Since Y is nearly-Lindelof, by Proposition 1 5 there exists $\bar{y} \in Y$ such that $\bar{y} \in f(C_\mu)$ for each $\mu \in M$ It follows that $f^{-1}(\bar{y}) \cap C_\mu \neq \emptyset$ for each $\mu \in M$, hence $f^{-1}(\bar{y})$ intersects all countable intersections of C_λ with $\lambda \in \Lambda$ Since $f^{-1}(\bar{y})$ is nearly-Lindelof relative to X, by Proposition 2 7 (iii), we have $\left(\underset{\lambda\in\Lambda}{\cap} C_\lambda\right) \cap f^{-1}(\bar{y}) \neq \emptyset$ and thus $\underset{\lambda\in\Lambda}{\cap} C_\lambda \neq \emptyset$ This, by Proposition 1 5, implies that X is nearly-Lindelof □

REMARK 2.10. Recall that, for a topological space X, the *Lindelof number* $l(X)$ is defined as the smallest cardinal number λ such that every open cover of X admits a subcover of cardinality λ It is natural to generalize this notion to nearly-Lindelof space defining the *nearly-Lindelof number of* X $nl(X)$ to be the smallest cardinal number μ such that every regularly open cover of X admits a subcover of cardinality μ

Obviously $nl(X) \leq l(X)$ and this inequality can be proper For this purpose we can see Example 1 2

§3. ALMOST-LINDELÖF AND WEAKLY-LINDELÖF SPACES

DEFINITION 3.1 [10] A topological space X is called *almost-Lindelöf* if every open cover $\{U_\lambda\}_{\lambda\in\Lambda}$ of X admits a countable subfamily such that $X = \underset{n\in\mathbf{N}}{\cup} \overline{U_{\lambda_n}}$

DEFINITION 3.2 [3] A topological space X is said to be *weakly-Lindelof* if for every open cover $\{U_\lambda\}_{\lambda\in\Lambda}$ of X there exists a countable subfamily such that $X = \overline{\underset{n\in\mathbf{N}}{\cup} U_{\lambda_n}}$

PROPOSITION 3.3. A topological space X is weakly-Lindelof if and only if for any family of closed subsets of $X\{C_\lambda\}_{\lambda\in\Lambda}$ such that $\underset{\lambda\in\Lambda}{\cap} C_\lambda = \emptyset$ there exists a countable subfamily $\{C_{\lambda_n}\}_{n\in\mathbf{N}}$ such that $\mathrm{int}\left(\underset{n\in\mathbf{N}}{\cap} C_{\lambda_n}\right) = \emptyset$.

PROOF. Let $\{C_\lambda\}_{\lambda\in\Lambda}$ be a family of closed subsets of X such that $\underset{\lambda\in\Lambda}{\cap} C_\lambda = \emptyset$ Then $X = \underset{\lambda\in\Lambda}{\cup} (X \backslash C_\lambda)$, so by hypothesis there exists a countable subfamily such that $X = \overline{\underset{n\in\mathbf{N}}{\cup} (X\backslash C_{\lambda_n})}$ Hence $X \backslash \overline{\underset{n\in\mathbf{N}}{\cup} (X\backslash C_{\lambda_n})} = \emptyset$, i e $\mathrm{int}\left(X\backslash\left(\underset{n\in\mathbf{N}}{\cup}(X\backslash C_{\lambda_n})\right)\right) = \mathrm{int}\left(\underset{n\in\mathbf{N}}{\cap} C_{\lambda_n}\right) = \emptyset$ Conversely, let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of X Then $\underset{\lambda\in\Lambda}{\cap} (X\backslash U_\lambda) = \emptyset$ and therefore there exists a countable subfamily such that $\mathrm{int}\left(\underset{n\in\mathbf{N}}{\cap}(X\backslash U_{\lambda_n})\right) = \emptyset$. So

$$X = X\backslash \mathrm{int}\left(\underset{n\in\mathbf{N}}{\cap}(X\backslash U_{\lambda_n})\right) = \overline{X\backslash\left(\underset{n\in\mathbf{N}}{\cap}(X\backslash U_{\lambda_n})\right)} = \overline{\underset{n\in\mathbf{N}}{\cup} U_{\lambda_n}}. \ \square$$

PROPOSITION 3.4. Let X be a topological space For the following conditions

(i) X is weakly-Lindelof,

(ii) any cover $\{U_\lambda\}_{\lambda\in\Lambda}$ of X by regularly open sets of X admits a countable subfamily with dense union in X,

(iii) if $\{C_\lambda\}_{\lambda\in\Lambda}$ is a family of regularly closed subsets of X such that $\bigcap_{\lambda\in\Lambda} C_\lambda = \emptyset$, then there exists a countable subfamily such that $\operatorname{int}\left(\bigcap_{n\in\mathbb{N}} C_{\lambda_n}\right) = \emptyset$,

we have that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (ii) and if X is semiregular then (ii) $\Rightarrow$ (i)

PROOF. (i) $\Rightarrow$ (ii) is obvious from the definition The proof of (ii) $\Leftrightarrow$ (iii) is quite similar to the proof of Proposition 3 3 replacing open cover with a regularly open cover of X We will prove the implication (ii) $\Rightarrow$ (i) when X is semiregular Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of X By hypothesis we can suppose any U_λ to be regularly open Then there exists a countable subfamily $\{U_{\lambda_n}\}_{n\in\mathbb{N}}$ such that $\overline{\bigcup_{n\in\mathbb{N}} U_{\lambda_n}} = X$ This completes the proof □

Obviously, if a space is nearly-Lindelof then it is almost-Lindelof and if a space is almost-Lindelof then it is weakly-Lindelof But the following example shows that weakly-Lindelof property or almost-Lindelof property does not imply the nearly-Lindelof property

EXAMPLE 3.5. Let Ω be the smallest uncountable ordinal number and $A = [0,\Omega)$ as in Example 1 2 Let $X = \{a_{ij}, b_{ij}, c_i, a, b\}$ where $i \in A$ and $j \in \mathbb{N}$ Consider in X the topology such that the points $\{a_{ij}\}$ and $\{b_{ij}\}$are isolated and the fundamental system of neighborhoods of the points $\{c_i\}$, $\{a\}$ and $\{b\}$ are

$$B^n_{c_i} = \{c_i, a_{ij}, b_{ij}\}_{j\geq n}, \quad B^\alpha_a = \{a, a_{ij}\}_{i\geq\alpha, j\in\mathbb{N}} \quad \text{and} \quad B^\alpha_b = \{b, b_{ij}\}_{i\geq\alpha, j\in\mathbb{N}}$$

respectively X so topologized is Hausdorff and semiregular but it is not nearly-Lindelof as we can see considering the regularly open cover $\left(\bigcup_{i\in A} B^0_{c_i}\right) \cup B^1_a \cup B^1_b$ But X is weakly-Lindelof Indeed, let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of X Then there exist $\lambda_1, \lambda_2 \in \Lambda$ such that $a \in U_{\lambda_1}$ and $b \in U_{\lambda_2}$ The set $X\backslash\overline{(U_{\lambda_1} \cup U_{\lambda_2})}$ is countable, so it follows easily that X is weakly-Lindelof. Note that this space X is also almost-Lindelof.

Below we will give the construction of an example of a weakly-Lindelof space that it is not almost-Lindelöf

PROPOSITION 3.6. A topological space X is almost-Lindelöf if and only if every family $\{C_\lambda\}_{\lambda\in\Lambda}$ of closed subsets of X such that $\bigcap_{\lambda\in\Lambda} C_\lambda = \emptyset$ admits a countable subfamily such that $\bigcap_{n\in\mathbb{N}} \mathring{C}_{\lambda_n} = \emptyset$

PROOF. If $\{C_\lambda\}_{\lambda\in\Lambda}$ is a family by closed subsets of X such that $\bigcap_{\lambda\in\Lambda} C_\lambda = \emptyset$, then the family $\{X\backslash C_\lambda\}_{\lambda\in\Lambda}$ is an open cover of X. By hypothesis there exists a countable subfamily such that $\bigcup_{n\in\mathbb{N}} \overline{X\backslash C_{\lambda_n}} = X$, i e. $\bigcap_{n\in\mathbb{N}} \mathring{C}_{\lambda_n} = \emptyset$. Conversely, let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of X Then $\{X\backslash U_\lambda\}_{\lambda\in\Lambda}$ is a family by closed sets such that $\bigcap_{\lambda\in\Lambda}(X\backslash U_\lambda) = \emptyset$ By hypothesis there exists a countable subfamily such that $\bigcap_{n\in\mathbb{N}} \operatorname{int}(X\backslash U_{\lambda_n}) = \emptyset$, i e $X = \bigcup_{n\in\mathbb{N}} \overline{U_{\lambda_n}}$ This completes the proof. □

PROPOSITION 3.7. Let X be a topological space For the following conditions

(i) X is almost-Lindelöf,

(ii) every regularly open cover $\{U_\lambda\}_{\lambda\in\Lambda}$ admits a countable subfamily such that $X = \bigcup_{n\in\mathbb{N}} \overline{U_{\lambda_n}}$,

(iii) every family $\{C_\lambda\}_{\lambda\in\Lambda}$ of regularly closed subsets of X such that $\bigcap_{\lambda\in\Lambda} C_\lambda = \emptyset$ admits a countable subfamily such that $\bigcap_{n\in\mathbb{N}} \mathring{C}_{\lambda_n} = \emptyset$;

we have that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (ii) and if X is semiregular then (ii) $\Rightarrow$ (i)

PROOF. (i) $\Rightarrow$ (ii) is obvious by the definition. The proof of (ii) $\Leftrightarrow$ (iii) is quite similar to the proof of Proposition 3 6 replacing open cover with a regularly open cover of X We will prove the implication (ii) $\Rightarrow$ (i) when X is semiregular. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of X. By hypothesis we can suppose that any U_λ is regularly open, then there exists a countable subfamily $\{U_{\lambda_n}\}_{n\in\mathbb{N}}$ such that $\bigcup_{n\in\mathbb{N}} \overline{U_{\lambda_n}} = X$ This completes the proof. □

THEOREM 3.8. A weakly-Lindelof, semiregular and nearly paracompact space X is almost-Lindelof

PROOF. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a cover of X by regularly open sets Since X is nearly paracompact, this cover admits a locally finite refinement $\{V_\gamma\}_{\gamma\in\Gamma}$ Since X is weakly-Lindelof then there exists a countable subfamily such that $X = \overline{\bigcup_{n\in\mathbb{N}} V_{\gamma_n}}$ Since the family $\{V_{\gamma_n}\}_{n\in\mathbb{N}}$ is locally finite, then $\overline{\bigcup_{n\in\mathbb{N}} V_{\gamma_n}} = \bigcup_{n\in\mathbb{N}} \overline{V_{\gamma_n}}$ [2, 1 1 11] Choosing, for each $n \in \mathbb{N}$, $\lambda_n \in \Lambda$ such that $V_{\gamma_n} \subset U_{\gamma_n}$, we obtain $X = \bigcup_{n\in\mathbb{N}} \overline{V_{\gamma_n}} = \bigcup_{n\in\mathbb{N}} \overline{U_{\gamma_n}}$ By Proposition 3 7 X is almost-Lindelof □

PROPOSITION 3.9 [9] An almost regular space is an almost-Lindelof space if and only if it is nearly-Lindelof □

CONSTRUCTION OF A WEAKLY-LINDELÖF SPACE

LEMMA 3.10. The real line $\mathbb{R}$ can be partitioned in the union of a family, of cardinality $2^{\aleph_0}$, by countable dense and pairwise disjoint subsets of $\mathbb{R}$

PROOF. Let $\mathbb{Q}$ be the set of the rational numbers Consider the following equivalence relation on $\mathbb{R}$

$$x \sim y \text{ if and only if } x - y \in \mathbb{Q}.$$

The partition of $\mathbb{R}$ so obtained is the one that we want, in fact every equivalence class is of the form $x + \mathbb{Q}$, where $x \in \mathbb{R}$, and it is a countable dense subset of $\mathbb{R}$ □

EXAMPLE 3.11. Let $\mathbb{R}$ be the real line and τ the usual topology on it By the previous lemma we can represent $\mathbb{R}$ as a union of continuum many countable dense and pairwise disjoint subsets of $\mathbb{R}$ We can write this partition as $\mathbb{R} = \left(\bigcup_{i\in I} S_i\right) \cup S_0$, where the set I has cardinality $2^{\aleph_0}$ Let τ_1 be the topology on $\mathbb{R}$ having the base $\{S_i\}_{i\in I} \cup \mathbb{R}$ Let σ be the topology generated by τ and τ_1 and let $X = (\mathbb{R}, \sigma)$ We will show that X is not almost-Lindelof Since S_0 is countable, we can write $S_0 = \{x_1, x_2, ..., x_n, ...\}$ Consider the open cover of X

$$X = \left(\bigcup_{i\in I} S_i\right) \cup \left(\bigcup_{n\in\mathbb{N}} \left]x_n - \frac{1}{n^2}, x_n + \frac{1}{n^2}\right[\right).$$

Suppose that X is almost-Lindelöf, then there exists a countable set $\{i_1, i_2, ..., i_n, ...\} \subset I$ such that

$$X = \left(\bigcup_{n\in\mathbb{N}} \overline{S_{i_n}}\right) \cup \left(\bigcup_{n\in\mathbb{N}} \left[x_n - \frac{1}{n^2}, x_n + \frac{1}{n^2}\right]\right) = \bigcup_{n\in\mathbb{N}} (S_{i_n} \cup S_0) \cup \left(\bigcup_{n\in\mathbb{N}} \left[x_n - \frac{1}{n^2}, x_n + \frac{1}{n^2}\right]\right).$$

Since the Lebesgue measure of the set $\bigcup_{n\in\mathbb{N}} [x_n - \frac{1}{n^2}, x_n + \frac{1}{n^2}]$ is finite, then $X \setminus \left(\bigcup_{n\in\mathbb{N}} [x_n - \frac{1}{n^2}, x_n + \frac{1}{n^2}]\right)$ has cardinality greater than $\aleph_0$ But $X \setminus \left(\bigcup_{n\in\mathbb{N}} [x_n - \frac{1}{n^2}, x_n + \frac{1}{n^2}]\right) \subset \bigcup_{n\in\mathbb{N}} (S_{i_n} \cup S_0)$ and, since the second member is countable, we obtain a contradiction. We will show now that X is weakly-Lindelof Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of X and $S_0 = \{x_0, x_1, ..., x_n, ...\}$ as above. Since in the topology σ every point of S_0 has the same fundamental system of neighborhoods as in the topology τ, then for each $n \in \mathbb{N}$ there exist an open set V_n in τ and an index $\lambda_n \in \Lambda$ such that $x_n \in V_n \subset U_{\lambda_n}$ The set $V = \bigcup_{n\in\mathbb{N}} V_n$ is open in τ and $S_0 \subset V \subset \bigcup_{n\in\mathbb{N}} U_{\lambda_n}$. Let $x_i \in S_i$. For any τ-open neighborhood V_i of x_i it is $V_i \cap S_0 = \emptyset$ (because S_0 is dense in $(\mathbb{R}, \tau)$) So $V_i \cap V \neq \emptyset$, hence $S_i \cap V_i \cap V \neq \emptyset$ and this shows that $x_i \in \mathrm{cl}_\sigma(V)$ We obtain that $X = \mathrm{cl}_\sigma(V) = \mathrm{cl}_\sigma\left(\bigcup_{n\in\mathbb{N}} U_{\lambda_n}\right)$ and therefore X is weakly-Lindelöf □

§4. ALMOST REGULAR-LINDELÖF SPACES

The previous example suggests some interesting remarks But before it is useful to recall the following definitions

DEFINITION 4.1 [1] An open cover $\{U_\lambda\}_{\lambda\in\Lambda}$ of a topological space X is said to be *regular* if for every $\lambda \in \Lambda$ there exists a non-empty regularly closed subset C_λ of X such that $C_\lambda \subseteq U_\lambda$ (i e U_λ is quasi regular open) and $\bigcup_{\lambda\in\Lambda} \overset{\circ}{C}_\lambda = X$

DEFINITION 4.2 [1] A topological space X is said to be *weakly compact* if every regular cover admits a finite subfamily such that the union is dense in X

LEMMA 4.3. Let X be the space in Example 3 11 Let C be a regularly closed and A an open set such that $C \subset A$ Then $\text{int}_\sigma(C) \subset \text{int}_\tau \text{cl}_\sigma(A)$

PROOF. We denote with x_0 and x_i the elements of S_0 and S_i respectively We show before that if $x_0 \in \text{int}_\sigma(C)$ then $x_0 \in \text{int}_\tau \text{cl}_\sigma(A)$ Since the fundamental system of neighborhoods of x_0 is the same whether in the topology σ or in τ, then the lemma is true Now let $x_i \in \text{int}_\sigma(C)$ There exists a τ-open neighborhood V_i of x_i such that $V_i \cap S_i \subset C$ We will show that $V_i \subset \text{cl}_\sigma(A)$ Let $x_0 \in V_i$ and let V_0 be an arbitrary σ-open, and therefore τ-open, neighborhood of x_0 Since $x_0 \in V_i \cap V_0$, we have $V_i \cap V_0 \neq \emptyset$ and thus $S_i \cap V_i \cap V_0 \neq \emptyset$ This shows that $x_0 \in \text{cl}_\sigma(V_i \cap S_i) \subset C \subset \text{cl}_\sigma(A)$ Let $x_j \in V_i$ Suppose that $x_j \notin \text{cl}_\sigma(A)$, i e there exists a τ-open neighborhood V_j of x_j such that $V_j \cap S_j \cap A = \emptyset$ Since $x_j \in V_j \cap V_i$, then $V_j \cap V_i \neq \emptyset$ and therefore $V_j \cap V_i \cap S_0 \neq \emptyset$ Let $x_0 \in V_j \cap V_i$ We have seen above that $x_0 \in C \subset A \subset \text{cl}_\sigma(A)$, since A is σ-open hence there exists a σ-open, and therefore τ-open neighborhood V_0 of x_0 such that $V_0 \subset A$ Since $V_0 \cap V_i \cap V_j \neq \emptyset$ then, by density of S_j, $V_0 \cap V_i \cap V_j \cap S_j \neq \emptyset$ and therefore $A \cap V_j \cap S_j \neq \emptyset$ that is a contradiction So it is shown that $x_j \in \text{cl}_\sigma(A)$ and therefore $V_i \subset \text{cl}_\sigma(A)$ The proof is complete □

PROPERTY 4.5. The space X in Example 3 11 satisfies the following property every regular cover $\{U_\lambda\}_{\lambda\in\Lambda}$ of X admits a countable subfamily $\{U_{\lambda_n}\}_{n\in\mathbb{N}}$ such that $X = \bigcup_{n\in\mathbb{N}} \overline{U_{\lambda_n}}$

PROOF. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a regular cover of X For any $\lambda \in \Lambda$ there exists a regularly closed $C_\lambda \subset U_\lambda$ such that $X = \bigcup_{\lambda\in\Lambda} \text{int}_\sigma(C_\lambda)$ By the previous lemma we have $X = \bigcup_{\lambda\in\Lambda} \text{int}_\tau \text{cl}_\sigma(U_\lambda)$ and, since X is Lindelof with respect to the topology τ, there exists a countable subcover such that $X = \bigcup_{n\in\mathbb{N}} \text{int}_\tau \text{cl}_\sigma(U_{\lambda_n}) = \bigcup_{n\in\mathbb{N}} \text{cl}_\sigma(U_{\lambda_n})$. □

The previous property suggests us to give a new definition that generalizes the weakly-Lindelof property

DEFINITION 4.6. A topological space X is called *almost regular-Lindelof* if every regular cover $\{U_\lambda\}_{\lambda\in\Lambda}$ of X admits a countable subfamily such that $X = \bigcup_{n\in N} \overline{U_{\lambda_n}}$

REMARK 4.7. Obviously almost-Lindelof implies almost regular-Lindelöf, but the converse in general is not true, in fact the space X in Example 3.11 is almost regular-Lindelof but not almost-Lindelof

THEOREM 4.8. An almost regular-Lindelof and almost regular space X is nearly-Lindelof

PROOF. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a cover by regularly open sets of X For each $x \in X$ there exists $\lambda_x \in \Lambda$ such that $x \in U_{\lambda_x}$ Since X is almost regular, there exist two regularly open subsets V_{λ_x} and W_{λ_x} such that $x \in V_{\lambda_x} \subset \overline{V_{\lambda_x}} \subset W_{\lambda_x} \overline{W}_{\lambda_x} \subset U_{\lambda_x}$ [13, Th 2 2] The family $\{W_{\lambda_x}\}_{x\in X}$ is a regular cover of X and, since X is almost regular-Lindelöf, there exists a countable set of points $x_1, x_2, ..., x_n, ...$ of X such that $X = \bigcup_{x\in\mathbf{N}} \overline{W_{\lambda_x}}$ So $X = \bigcup_{n\in\mathbb{N}} U_{\lambda_{xn}}$ and therefore X is nearly-Lindelof □

The previous theorem implies the following

COROLLARY 4.9. Let X be an almost regular space Then X is almost regular-Lindelof if and only if it is nearly-Lindelof □

We give now a characterization of almost regular-Lindelof spaces

THEOREM 4.10. A topological space X is almost regular-Lindelof if and only if for every family $\{C_\lambda\}_{\lambda\in\Lambda}$ of closed subsets of X, such that for each $\lambda \in \Lambda$ there exists an open set $A_\lambda \supset C_\lambda$ with $\bigcap_{\lambda\in\Lambda} \overline{A_\lambda} = \emptyset$, there exists a countable subfamily such that $\bigcap_{n\in\mathbb{N}} \mathring{C}_{\lambda_n} = \emptyset$

PROOF. Let $\{C_\lambda\}_{\lambda\in\Lambda}$ be a family of closed sets of X such that for each $\lambda \in \Lambda$ there exists an open set $A_\lambda \supset C_\lambda$ with $\bigcap_{\lambda\in\Lambda} \overline{A_\lambda} = \emptyset$ It follows that $X = X \backslash \left(\bigcap_{\lambda\in\Lambda} \overline{A_\lambda} \right) = \bigcup_{\lambda\in\Lambda} (X \backslash \overline{A_\lambda})$ But, since $C_\lambda \subset A_\lambda \subset \mathring{\overline{A_\lambda}} \subset \overline{A_\lambda}$, then $X \backslash \overline{A_\lambda} \subset X \backslash \mathring{\overline{A_\lambda}} \subset X \backslash C_\lambda$, and therefore $X = \bigcup_{\lambda\in\Lambda} (X \backslash C_\lambda)$ The family

$\{X\backslash C_\lambda\}_{\lambda\in\Lambda}$ is a regular cover of X, since X is almost regular-Lindelof, then there exists a countable subfamily such that

$$X = \bigcup_{n\in\mathbb{N}} (\overline{X\backslash C_{\lambda_n}}) = \bigcup_{n\in\mathbb{N}} \left(X\backslash \mathring{C}_{\lambda_n}\right) = X\backslash\left(\bigcap_{n\in\mathbb{N}} \mathring{C}_{\lambda_n}\right)$$

and therefore $\bigcap_{n\in\mathbb{N}} \mathring{C}_{\lambda_n} = \emptyset$ Conversely, let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a regular cover of X For each $\lambda\in\Lambda$ there exists a regularly closed C_λ of X such that $\mathring{C}_\lambda \subset C_\lambda \subset U_\lambda$ and $\bigcup_{\lambda\in\Lambda} \mathring{C}_\lambda = X$ The family $\{X\backslash U_\lambda\}_{\lambda\in\Lambda}$ of closed sets is such that, for each $\lambda\in\Lambda$, there exists the open set $X\backslash C_\lambda \supset X\backslash U_\lambda$ and such that

$$\bigcap_{\lambda\in\Lambda} \overline{X\backslash C_\lambda} = \bigcap_{\lambda\in\Lambda}\left(X\backslash \mathring{C}_\lambda\right) = X\backslash\left(\bigcup_{\lambda\in\Lambda}\mathring{C}_\lambda\right) = X\backslash X = \emptyset.$$

By hypothesis, there exists a countable set of indices $\{\lambda_n\}_{n\in\mathbb{N}}$ such that $\bigcap_{n\in\mathbb{N}} \mathrm{int}(X\backslash U_{\lambda_n}) = \emptyset$, i e $\bigcap_{n\in\mathbb{N}} (X\backslash \overline{U}_{\lambda_n}) = \emptyset$ So $X\backslash\left(\bigcup_{n\in\mathbb{N}} \overline{U_{\lambda_n}}\right) = \emptyset$ and therefore $X = \bigcup_{n\in\mathbb{N}} \overline{U_{\lambda_n}}$ This completes the proof □

ALMOST REGULAR-LINDELÖF SUBSPACES AND SUBSETS

A subset S of a space X is said to be almost regular-Lindelof if S is almost regular-Lindelof as a subspace of X

DEFINITION 4.11. A subset S of a space X is said to be *almost regular-Lindelof relative to X* if for each family $\{U_\lambda\}_{\lambda\in\Lambda}$ of open sets of X satisfying the condition

$$S \subset \bigcup_{\lambda\in\Lambda} U_\lambda, \text{ and}$$

(*) for each $\lambda\in\Lambda$, there exists a nonempty regularly closed set C_λ of X such that $C_\lambda \subset U_\lambda$ and $S \subset \bigcup_{\lambda\in\Lambda} \mathring{C}_\lambda$,

there exists a countable subset $\{\lambda_n\}_{n\in\mathbb{N}} \subset A$ such that $S \subset \bigcup_{n\in\mathbb{N}} \overline{U_{\lambda_n}}$

THEOREM 4.12. If S is an almost regular-Lindelöf subspace of a space X, then S is almost regular-Lindelof relative to X

PROOF. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a cover of S satisfying the condition (*) For each $\lambda\in\Lambda$, we have that $\mathring{C}_\lambda\cap S$ and $U_\lambda \cap S$ are open sets in S, and $C_\lambda\cap S$ is closed in S The family $\{U_\lambda\cap S\}_{\lambda\in\Lambda}$ is an open cover of S We will show that it is a regular cover of the subspace S For each $\lambda\in\Lambda$, we have that $\mathrm{cl}_S\left(\mathring{C}_\lambda\cap S\right) \subset C_\lambda\cap S\subset U_\lambda\cap S$, where $\mathrm{cl}_S\left(\mathring{C}_\lambda\cap S\right)$ is regularly closed in S Moreover, we have $S = \bigcap_{\lambda\in\Lambda}\left(\mathring{C}_\lambda\cap S\right)$ and $\mathring{C}_\lambda\cap S \subset \mathrm{int}_S\mathrm{cl}_S\left(\mathring{C}_\lambda\cap S\right)$, thus $S = \bigcup_{\lambda\in\Lambda}\mathrm{int}_S\mathrm{cl}_S\left(\mathring{C}_\lambda\cap S\right)$. Since S is an almost regular-Lindelöf subspace of X, there exists a countable subcover such that $S = \bigcup_{n\in\mathbb{N}} \mathrm{cl}_S(U_{\lambda_n}\cap S)$ Since for each $n\in\mathbb{N}$ $\mathrm{cl}_S(U_{\lambda_n}\cap S)\subset\overline{U_{\lambda_n}}$, we obtain that $S\subset\bigcup_{n\in\mathbb{N}}\overline{U_{\lambda_n}}$. This shows that S is almost regular-Lindelöf relative to X □

THEOREM 4.13. If every regularly closed subset of a space X is almost regular-Lindelof relative to X, then X is almost regular-Lindelöf

PROOF. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a regular cover of X. For each $\lambda\in\Lambda$, there exists a nonempty regularly closed set C_λ of X such that $C_\lambda\subset U_\lambda$ and $X = \bigcup_{\lambda\in\Lambda}\mathring{C}_\lambda$ Fix an arbitrary $\lambda_0\in\Lambda$ and let $\Lambda^* = \Lambda\backslash\{\lambda_0\}$ Put $K = X\backslash\mathring{C}_{\lambda_0}$, then K is regularly closed in X and $K\subset\bigcup_{\lambda\in\Lambda^*}\mathring{C}_\lambda$. Therefore $\{U_\lambda\}_{\lambda\in\Lambda^*}$ is a cover of K by open sets of X satisfying the condition (*) of Definition 4 11 and hence there exists a countable subset $\{\lambda_n\}_{n\in\mathbb{N}}\subset\Lambda^*$ such that $K\subset\bigcup_{n\in\mathbb{N}^*}\overline{U_{\lambda_n}}$ So we have

$$X = K\cup\mathring{C}_{\lambda_0} = K\cup\overline{U_{\lambda_0}} = \left(\bigcup_{n\in\mathbb{N}^*}\overline{U_{\lambda_n}}\right)\cup\overline{U_{\lambda_0}} = \bigcup_{n\in\mathbb{N}}\overline{U_{\lambda_n}}.$$

This shows that X is almost regular-Lindelof □

COROLLARY 4.14. If every proper regularly closed subset of a space X is almost regular-Lindelof, then X is almost regular-Lindelof. □

THEOREM 4.15. Let X be an almost regular-Lindelof space If A is a proper clopen subset of X, then A is almost regular-Lindelöf relative to X

PROOF. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a cover of A by open sets of X satisfying the condition (*) of Definition 4 11 The family $\{U_\lambda\}_{\lambda\in\Lambda} \cup (X\backslash A)$ is a regular cover of X Since X is almost regular-Lindelof, there exists a countable subfamily such that

$$X = \left(\bigcup_{n\in\mathbb{N}} \overline{U_{\lambda_n}}\right) \cup \left(\overline{X\backslash A}\right) = \left(\bigcup_{n\in\mathbb{N}} \overline{U_{\lambda_n}}\right) \cup (X\backslash A).$$

Therefore we have $A \subset \bigcup_{n\in\mathbb{N}} \overline{U_{\lambda_n}}$ This completes the proof □

We conclude this paper introducing the following two definitions

DEFINITION 4.16. A space X is called *weakly regular-Lindelof* if every regular cover $\{U_\lambda\}_{\lambda\in\Lambda}$ of X admits a countable subfamily such that $X = \overline{\bigcup_{n\in\mathbb{N}} U_{\lambda_n}}$

DEFINITION 4.17. A space X is called *nearly regular-Lindelof* if every regular cover $\{U_\lambda\}_{\lambda\in\Lambda}$ of X admits a countable subfamily such that $X = \bigcup_{n\in\mathbb{N}} \overset{\circ}{\overline{U_{\lambda_n}}}$

Obviously we have the following implications

$$\begin{array}{ccccc} \text{Nearly-L} & \Rightarrow & \text{Almost-L} & \Rightarrow & \text{Weakly-L} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{Nearly regular-L} & \Rightarrow & \text{Almost regular-L} & \Rightarrow & \text{Weakly regular-L} \end{array}$$

We leave open the study of these two new generalizations of Lindelöf property and the relative implications

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