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A NOTE ON RINGS OF CONTINUOUS FUNCTIONS

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ABSTRACT. For a topological space X, and a topological ring A, let C(X,A) be the ring of all continuous functions from X into A under the pointwise multiplication. We show that the theorem "there is a completely regular space Y associated with a given topological space X such that C(Y,R) is isomorphic to C(X,R)" may be extended to a fairly large class of topological rings, and that, in the study of algebraic structure of the ring C(X,A), it is sufficient to study C(X,R) if A is path connected.

For a topological space X and a topological ring A, let C(X,A) denote the ring of all continuous functions from X into A under the pointwise multiplication. If A is the ring of real numbers R with the usual topology, C(X,R) will simply be denoted by C(X). In [2], the structure of the ring C(X,A), where X is totally disconnected, is studied. The topologies for

C(X,A) considered there are the topology of pointwise convergence, the compact-open topology, and the topology of uniform convergence. Under each of these topologies C(X,A) is a topological ring.

In the study of rings of real-valued continuous functions on a topological space, it is usually assumed that X is completely regular. This assumption of complete regularity on X has no loss of generality as it can be seen in the following.

THEOREM 1. For every topological space X, there exists a completely regular space Y such that C(Y) is (algebraically) isomorphic to C(X).

The purpose of this note is to show that the above theorem may be extended to fairly large class of topological rings A, and that, in the study of algebraic structure of the ring C(X,A), it is sufficient to study C(X) if A is path connected. All topological spaces considered here are assumed to be Hausdorff.

The following definition is a modification of the one given in [4].

DEFINITION A pair (X,A) of a topological space X and a topological ring A is called an S-pair, if for each closed subset C of X and $x \notin C$, there exists $f \in C(X,A)$ such that $f(x) \neq 0$ and $Z(f) = \{x \mid f(x) = 0\} \supset C$, where 0 is the zero element of the ring A.

It is easy to see that if X is completely regular and A is path connected, or if X is 0-dimensional and A is any topological ring, then (X,A) is an S-pair.

REMARK If $\{(X_\alpha, A_\alpha): \alpha \in I\}$ is a family of S-pairs, then $(\Pi_{\alpha\in I} X_\alpha, \Pi_{\alpha\in I} A_\alpha)$ is also an S-pair, where $\Pi_{\alpha\in I} X_\alpha$ denotes the product space of the space X while $\Pi_{\alpha\in I} A_\alpha$ denoted the direct product of the rings A_α.

PROOF: Let C be a closed subset of $X = \Pi_{\alpha\in I} X_\alpha$, and let $x \notin C$. Then there exists some basic neighborhood of x

$$\Pi_{\alpha_1}^{-1}(U_1) \cap \Pi_{\alpha_2}^{-1}(U_2) \cap \ldots \cap \Pi_{\alpha_n}^{-1}(U_n)$$

which is disjoint from C, where each U_i is open in X_{α_i}, $i = 1,2,\ldots,n$. For each i, $i = 1,2,\ldots,n$, let $f_i \in C(X_{\alpha_i}, A_{\alpha_i})$ such that $Z(f_i) \supset X_{\alpha_i} - U_i$, and $f(x_{\alpha_i}) \neq 0_{\alpha_i}$, where $x = (x_\alpha)$. Define g: $X \to A = \prod_{\alpha\in I} A_\alpha$ as follows: For $y = (y_\alpha) \in X$, let $g(y) = (t_\alpha)$ where $t_\alpha = f_i(y_{\alpha_i})$ if $\alpha = \alpha_i$ and $t_\alpha = 0_\alpha$ if $\alpha \neq \alpha_i$ for $i = 1,2,\ldots,n$. Then $g \in C(X,A)$, $Z(g) \supset C$, and $g(x) \neq 0$.

A topological space X is called a V-space, [3], if for points p, q, x, and y of X, where $p \neq q$, there exists a continuous functions f of X into itself such that $f(p) = x$ and $f(q) = y$. It is shown [3] that every completely regular path connected space and every zero-dimensional space is a V-space. It is easy to see that if (x_α, A) is an S-pair for each $\alpha \in I$, then $(\prod_{\alpha\in I} X, A)$ is also an S-pair if the underlying space of A is a V-space. One may ask the question that if A is a topological ring such that (A,A) is an S-pair, is A a V-space? The answer to this question is negative as the following example shows.

EXAMPLE 1. Let R_1 be the ring of real numbers with the usual topology, and let R_2 be the ring of integers with the discrete topology. Then R_1 is path connected while R_2 is zero-dimensional, thus (R_1, R_1) and (R_2, R_2) are S-pairs. Hence $(R_1 \times R_2, R_1 \times R_2)$ is also an S-pair by the remark above. Since $R_1 \times R_2$ is not connected with all components homeomorphic to R_1, it follows from Theorem 3.5 of [3] that $R_1 \times R_2$ is not a V-space.

Now let X be a topological space, and A be a topological ring. For x and y in X, define $x \equiv_A y$ if and only if $f(x) = f(y)$ for each $f \in C(X,A)$. Then "$\equiv_A$" is an equivalence relation in X. Let Y_A be the set of all equivalence classes, and let T: $X \to Y_A$ be the natural map. For each $f \in C(X,A)$, let f_T: $Y_A \to A$ be defined by $f_T([x]) = f(x)$. Then f_T is welldefined, and $f_T \circ T = f$ for each $f \in C(X,A)$.

$$\text{Let } C_A = \{f_T \in A^{Y_A} \mid f \in C(X,A)\}$$
$$= \{g \in A^{Y_A} \mid g \circ T \in C(X,A)\}$$

and let τ_A be the weak topology on Y_A induced by the family C_A. Note that the construction of the space Y_A is analogous to that of the space Y of Theorem 1.

THEOREM 2 (1) The topological space (Y_A,τ_A) is Hausdorff.

(2) (Y_A,τ_A) is completely regular.

(3) The mapping T: $X \to (Y_A,\tau_A)$ is continuous.

(4) The mapping ϕ: $g \to g \circ T$ of $C(Y_A,A)$ onto $C(X,A)$ is a continuous isomorphism, where $C(Z,A)$ is assumed to have the compact-open topology.

PROOF: (3) and (4) are clear.

To show (1), let $y_1, y_2 \in Y_A$, where $y_1 = [x_1]$, $y_2 = [x_2]$, and $y_1 \neq y_2$. Then there exists $f \in C(X,A)$ such that $f(x_1) \neq f(x_2)$. Thus $f_T(y_1) \neq f_T(y_2)$. If V_1 and V_2 are open sets in A such that $f_T(y_i) \in V_i$ for $i = 1,2$, and $V_1 \cap V_2 = \phi$, then $f_T^{-1}(V_1) \cap f_T^{-1}(V_2) = \phi$. Hence (Y_A,τ_A) is Hausdorff.

For (2), let $x \in U = f_{T_1}^{-1}(V_1) \cap f_{T_2}^{-1}(V_2) \cap \ldots \cap f_{T_n}^{-1}(V_n)$, where each V_i is open in A, and $f_{T_i} \in C_A$, $i = 1,2,\ldots,n$. Then $f_{T_i}(x) \in V_i$ for each $i = 1,2,\ldots,n$. For each i, there exists $g_i \in C(A,[0,1])$ such that $g_i(f_{T_i}(x)) \neq 0$ and $g_i(A-V_i) = 0$. If we let $h = (g_1 \circ f_{T_1})(g_2 \circ f_{T_2})\ldots(g_n \circ f_{T_n})$, then $h \in C(Y_A)$, and $h(x) \neq 0$ but $h(y) = 0$ for $y \notin U$. Hence (Y_A,τ_A) is completely regular.

It is noted that the map T need not be a quotient map as the following example, [1], shows.

EXAMPLE 2. Let S denote the subspace of R^2 obtained by deleting (0,0) and all points $(\frac{1}{n},y)$ with $y \neq 0$ and $n \in N$. Define $\pi(x,y) = x$ for all $(x,y) \in S$.

Let X be the quotient space of S induced by the mapping π then X can be identified as the set of real numbers endowed with the largest topology for which the mapping π is continuous. It is demonstrated in [1] that X is Hausdorff, not completely regular, Y_R is the space of real numbers, and that the mapping T is not a quotient map.

THEOREM 3 If the ring A is path connected, then

(1) $((Y_A,\tau_A),A)$ is an S-pair

(2) $Y_A = Y_R$

(3) $\tau_A = \tau_R$.

PROOF: Since A is assumed to be path connected while (Y_A,τ_A) is completely regular by Theorem 2, (1) is clear.

To show (2) it is sufficient to show that $x \equiv {}_R y$ if and only if $x \equiv {}_A y$ whenever $x, y \in X$. Let $x \equiv {}_R y$ but suppose that $x \not\equiv {}_A y$. Then there exists $f \in C(X,A)$ such that $f(x) \neq f(y)$. Let $g \in C(A)$ such that $g \circ f(x) \neq g \circ f(y)$. This would imply that $x \not\equiv {}_R y$ since $g \circ f \in C(X)$, a contradiction. Conversely, if $x \equiv {}_A y$ but $x \not\equiv {}_R y$. Then there exists $f \in C(X)$ such that $f(x) \neq f(y)$. Then there exists $h \in C(R,A)$ such that $h(f(x)) = 0$ but $h(f(y)) = t \neq 0$. If $g = h \circ f$, then $g \in C(X,A)$ but $g(x) \neq g(y)$ which leads to a contradiction again.

Finally we shall prove (3). Since A is completely regular C(A) separates points from closed sets in A, thus sets of the form $k^{-1}(V)$, where $k \in C(A)$ and V open in R, form a subbase for the topology of A. Let $f_T^{-1}(U)$ be a subbasic open set in τ_A. Then U is open in A, hence we may let $U = \bigcap_{i=1}^{n} k_i^{-1}(V_i)$, where for each $i = 1,2,3,\ldots,n$, $k_i \in C(A)$ and V_i open in R. Thus $f_T^{-1}(U) = f_T^{-1}(\bigcap_{i=1}^{n} k_i^{-1}(V_i)) = \bigcap_{i=1}^{n} f_T^{-1}(k_i^{-1}(V_i)) = \bigcap_{i=1}^{n} (k_i \circ f_T)^{-1}(V_i)$. Since $k_i \circ f_T \in R^{Y_A}$ and $Y_A = Y_R$ by (2), this implies that $\tau_A \subset \tau_R$. Conversely, let

$h^{-1}(U)$ be a subbasic open set in τ_R, where U is open in R and $h \circ T \in C(X)$. Let $y \in h^{-1}(U)$. Since (Y_R,τ_R) is completely regular, $((Y_R,\tau_R),A)$ is an S-pair, hence there exists $f \in C(Y_R,A)$ such that $f(y) \neq 0$ but $f(Y_R - h^{-1}(U)) = 0$. Then $y \in f^{-1}(A - \{0\}) \subset h^{-1}(U)$. Since $f \circ T \in C(X,A)$ and $f \in A^{Y_A}$, this shows that $h^{-1}(U) \in \tau_A$. Hence $\tau_R \subset \tau_A$.

The above theorem shows that, within the category of path connected topological rings, the space Y_A is independent of the ring A.

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