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# COMPARISON THEOREMS FOR ULTRAHYPERBOLIC EQUATIONS

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ABSTRACT. Sturmian comparison theorems are obtained for solutions of a class of normal and singular ultrahyperbolic partial differential equations. In the singular case, solutions are considered which satisfy non-standard boundary conditions.

KEY WORDS AND PHRASES. *Comparison theorems, ultrahyperbolic equations, singular boundary value problems, EDP equation.*

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1. Introduction. The Sturmian comparison theorems for ordinary differential equations have been extended extensively to partial differential equations of the elliptic type. For example, see Kuks [1], Swanson [2], [3], Diaz and McLaughlin [4], and Kreith and Travis [5], to mention only a few. By employing Swanson's technique, Dunninger [6] obtained a comparison theorem for parabolic partial differential equations. His result was recently generalized by Chan and Young [7] to time-dependent quasilinear differential systems. However, for partial differential equations of the hyperbolic type, very little is known. In fact, as far as these authors know, the only comparison results known for hyperbolic equations were obtained just recently by Kreith [8], Travis [9], [10], and Young [11].

In this paper, we shall present some comparison theorems for the pair of normal ultrahyperbolic equations

$$(A_{ij}(x)u_{x_i})_{x_j} - (a_{ij}(y)u_{y_i})_{y_j} + p(y)u = 0$$

$$(B_{ij}(x)v_{x_i})_{x_j} - (b_{ij}(y)v_{y_i})_{y_j} + q(y)v = 0$$

and the pair of singular ultrahyperbolic equations

$$(u_{x_i x_i} + \frac{\alpha_i}{x_i} u_{x_i}) - (a_{ij}(y)u_{y_i})_{y_j} + p(y)u = 0$$

$$(v_{x_i x_i} + \frac{\alpha_i}{x_i} v_{x_i}) - (b_{ij}(y)v_{y_i})_{y_j} + q(y)v = 0$$

in a domain $D = H \times G$, where $H$ and $G$ are bounded regular domains in $E^n$ and $E^m$, respectively. For brevity, we let $x = (x_1,\ldots,x_n)$ denote a point in $H$ and $y = (y_1,\ldots,y_m)$ a point in $G$. Moreover, we adopt Einstein's summation convention concerning repeated indices.

The matrices $(A_{ij})$, $(a_{ij})$, $(B_{ij})$ and $(b_{ij})$ are all assumed to be symmetric and positive definite with continuously differentiable elements in their respective domains of definition. The coefficients $p$ and $q$ are continuous in $G$, while the $\alpha_i$'s are real parameters, $-\infty < \alpha_i < \infty$, $i = 1,\ldots,n$.

We associate with the above equations the following eigenvalue problems

$$(1.1)\qquad \begin{aligned} &-(a_{ij}\phi_{y_i})_{y_j} + p\phi = \lambda\phi \quad \text{in} \quad G \\ &\frac{\partial\phi}{\partial n_a} + r(y)\phi = 0 \quad \text{on} \quad \partial G \end{aligned}$$

and

$$(1.2)\qquad \begin{aligned} &-(b_{ij}\psi_{y_i})_{y_j} + q\psi = \mu\psi \quad \text{in} \quad G \\ &\frac{\partial\psi}{\partial n_b} + s(y)\psi = 0 \quad \text{on} \quad \partial G \end{aligned}$$

where $r$ and $s$ are continuous functions on $\partial G$, and

$$\frac{\partial\phi}{\partial n_a} = a_{ij}\phi_{y_i}\nu_j, \quad \frac{\partial\phi}{\partial n_b} = b_{ij}\psi_{y_i}\nu_j,$$

$(\nu_1,\ldots,\nu_m)$ being the outward unit normal vector on $\partial G$.

2. The Normal Case. We consider the following boundary value problems

$$(2.1)\qquad \begin{aligned} &(A_{ij}u_{x_i})_{x_j} - (a_{ij}u_{y_i})_{y_j} + pu = 0 \quad \text{in} \quad D \\ &\frac{\partial u}{\partial n_a} + ru = 0 \quad \text{on} \quad H \times \partial G \end{aligned}$$

and

$$(2.2)\qquad \begin{aligned} &(B_{ij}v_{x_i})_{x_j} - (b_{ij}v_{y_i})_{y_j} + qv = 0 \quad \text{in} \quad D \\ &\frac{\partial v}{\partial n_b} + sv = 0 \quad \text{on} \quad H \times \partial G. \end{aligned}$$

<u>Theorem</u> 2.1. Let $u \in C^2(D)$ be a solution of (2.1) such that $u > 0$ in $D$, $u = 0$ on $\partial H \times G$. If

$$(2.3)\qquad (B_{ij}) \le (A_{ij}),\ (a_{ij}) \le (b_{ij}),\ p \le q,\ r \le s,$$

where at least one strict inequality holds, then every solution $v \in C^2(D)$ of (2.2) has a zero in $D$.

<u>Proof</u>. Suppose $v$ is a solution of (2.2). Let $\phi_0$ and $\psi_0$ be the positive eigenfunctions [12] corresponding to the first eigenvalues $\lambda_0$ and $\mu_0$ of (1.1) and (1.2), respectively. Define

$$U(x) = \int_G u(x,y)\phi_0(y)dy$$

$$V(x) = \int_G v(x,y)\psi_0(y)dy.$$

Since $u$ and $v$ satisfy (2.1) and (2.2), respectively, and $\phi_0$ and $\psi_0$ are eigenfunctions of (1.1) and (1.2), respectively, it follows by the divergence theorem that

$$(2.4)\qquad \begin{aligned} &(A_{ij}U_{x_i})_{x_j} + \lambda_0 U = 0 \quad \text{in} \quad H \\ &U = 0 \quad \text{in} \quad \partial H \end{aligned}$$

and

$$(2.5) \qquad (B_{ij}V_{x_i})_{x_j} + \mu_0 V = 0 \quad \text{in} \quad H.$$

By the variational characterization of the eigenvalues of (1.1) and (1.2), it follows from the assumptions (2.3) that $\lambda_0 < \mu_0$. Hence, by Theorem 10 of [4], $V$ has a zero in $H$, say at $x = x_0$. Then, since $\psi_0 > 0$ in $G$, the equation

$$V(x_0) = \int_G v(x_0,y)\psi_0(y)dy = 0$$

implies that $v$ must vanish at some point $y = y_0$ in $G$. Thus $(x_0,y_0)$ is a zero of $v$ in $D$, and the theorem is proved.

We remark that if the solution $u$ of (2.1) is required to satisfy the boundary condition

$$A_{ij}u_{x_i}t_j + R(x)u = 0$$

on $\partial H \times G$, instead of $u = 0$, the conclusion of Theorem 2.1 remains valid for solutions $v$ of (2.2) satisfying the additional condition

$$B_{ij}v_{x_i}t_j + S(x)v = 0$$

on $\partial H \times G$, where $R(x) \le S(x)$ and $(t_1,\ldots,t_n)$ is the outward unit normal vector on $\partial H$. This is a consequence of Theorem 2 of [3].

3. <u>The Singular Case</u>. We consider the singular boundary value problems

$$(3.1)\quad \begin{aligned} &U_{x_i x_i} + \frac{\alpha_i}{x_i} u_{x_i} - (a_{ij} u_{y_i})_{y_j} + pu = 0 \quad \text{in} \quad D \\ &\frac{\partial u}{\partial n_a} + ru = 0 \quad \text{on} \quad H \times \partial G \end{aligned}$$

and

$$(3.2)\quad \begin{aligned} &v_{x_i x_i} + \frac{\alpha_i}{x_i} v_{x_i} - (b_{ij} v_{y_i})_{y_j} + qv = 0 \quad \text{in} \quad D \\ &\frac{\partial v}{\partial n_b} + sv = 0 \quad \text{on} \quad H \times \partial G \end{aligned}$$

where now $H$ is the domain

$$H = \{x \in E^n \,|\, 0 < x_i < a_i,\ 1 \le i \le n\}.$$

Theorem 3.1. Let $\alpha_i \le -1$, $i = 1,\ldots,n$, and assume that

$$(3.3)\quad (a_{ij}) \le (b_{ij}),\ p \le q,\ r \le s \quad \text{in} \quad G.$$

If $u \in C^2(D)$ is a solution of (3.1) which is positive in $D$ and satisfies the conditions

$$(3.4)\quad \begin{aligned} &\int_0^a \int_G x^\alpha |u(x,y)|^2 dydx < \infty \\ &u(x,y) = 0 \quad \text{on} \quad x_i = a_i (1 \le i \le n) \end{aligned}$$

then every solution $v \in C^2(D)$ of (3.2) satisfying

$$(3.5)\quad \int_0^a \int_G x^\alpha |v(x,y)|^2 dydx < \infty$$

has a zero in $D$.

Here

$$\int_0^a \int_G x^\alpha |v|^2 dydx$$

denotes the integral

$$\int_0^{a_1} \cdots \int_0^{a_n} \int_G x_1^{\alpha_1} \cdots x_n^{\alpha_n} |v|^2 dydx$$

where as usual $dx = dx_1 \ldots dx_n$ and $dy = dy_1 \ldots dy_m$.

Proof. As in the proof of Theorem 2.1, we let $\phi_0$ and $\psi_0$ be the positive, normalized, eigenfunctions of (1.1) and (1.2) corresponding to the first eigenvalues $\lambda_0$ and $\mu_0$. Then the functions U and V, defined as in the proof of Theorem 2.1, satisfy the equations

$$\begin{aligned} &(x^\alpha U_{x_i})_{x_i} + \lambda_0 x^\alpha U = 0 \quad \text{in} \quad H \\ (3.6) \quad &\int_0^a x^\alpha |U(x)|^2 dx < \infty \\ &U(x) = 0 \quad \text{on} \quad x_i = a_i, \ (1 \le i \le n) \end{aligned}$$

and

$$\begin{aligned} &(x^\alpha V_{x_i})_{x_i} + \mu_0 x^\alpha V = 0 \quad \text{in} \quad H \\ (3.7) \quad &\int_0^a x^\alpha |V(x)|^2 dx < \infty. \end{aligned}$$

It can be shown, [13], that for $\alpha_i \le -1$, $i = 1,\ldots,n$, the only solutions of (3.6) and (3.7) are given by

$$U(x) = \prod_{i=1}^n x^{(1-\alpha_i)/2} J_{(1-\alpha_i)/2}(\sqrt{\lambda}_i x_i)$$

and

$$V(x) = \prod_{i=1}^{n} x_i^{(1-\alpha_i)/2} J_{(1-\alpha_i)/2}(\sqrt{\mu_i}\, x_i)$$

where $\lambda_0 = \lambda_1 + \ldots + \lambda_n$ and $\mu_0 = \mu_1 + \ldots + \mu_n$, the $\lambda_i$'s being the roots of the equation

$$J_{(1-\alpha_i)/2}(\sqrt{\lambda_i}\, a_i) = 0, \quad (i = 1,\ldots,n)$$

and the $\mu_i$'s are arbitrary constants. Here $J_p(t)$ denotes the Bessel function of the first kind of order $p$.

From condition (3.3), it follows that $\lambda_0 < \mu_0$. Hence there exists an integer $j$ such that $\lambda_j < \mu_j$. This implies that $V$ vanishes along the line $x_j = \sqrt{\lambda_j/\mu_j}\, a_j < a_j$. Since $\psi_0(y) > 0$ in $G$, we conclude by the same argument as in the proof of Theorem 2.1 that $V$ has a zero in $D$.

Theorem 3.2. Let $\alpha_i > -1$, $i = 1,\ldots,n$, and assume that condition (3.3) holds. If $u$ is a solution of (3.1) which is positive in $D$ and satisfies the conditions

$$\lim_{x_i \to 0} x_i^{\alpha_i} u_{x_i} = 0, \quad (i = 1,\ldots,n)$$

$$u(x,y) = 0, \quad \text{on } x_i = a_i, \quad (i = 1,\ldots,n).$$

then every solution $v$ of (3.2) satisfying

$$\lim_{x_i \to 0} x_i^{\alpha_i} v_{x_i} = 0, \quad (i = 1,\ldots,n)$$

has a zero in $D$.

The proof of this theorem is similar to that of Theorem 3.1 and is therefore omitted.

We conclude this paper with a theorem that is valid for all values of the parameters $\alpha_i$.

Theorem 3.3. Let $-\infty < \alpha_i < \infty$, $i = 1,\dots,n$, and assume that (3.3) holds. If $u$ is a solution of (3.1) which is positive in $D$ and satisfies the conditions

$$u(x,y) = 0 \quad \text{on} \quad x_i = a_i, \; (i = 1,\dots,n)$$

$$|u(0,y)| < \infty \quad \text{for} \quad y \in G$$

then every solution $v$ of (3.2) satisfying

$$|v(0,y)| < \infty, \; y \in G$$

has a zero in $D$.

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