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ON GENERATION AND PROPAGATION OF TSUNAMIS IN A SHALLOW RUNNING OCEAN

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ABSTRACT. A theory is presented of the generation and propagation of the two and the three dimensional tsunamis in a shallow running ocean due to the action of an arbitrary ocean floor or ocean surface disturbance. Integral solutions for both two and three dimensional problems are obtained by using the generalized Fourier and Laplace transforms. An asymptotic analysis is carried out for the investigation of the principal features of the free surface elevation. It is found that the propagation of the tsunamis depends on the relative magnitude of the given speed of the running ocean and the wave speed of the shallow ocean. When the speed of the running ocean is less than the speed of the shallow ocean wave, both the two and the three dimensional free surface elevation represent the generation and propagation of surface waves which decay asymptotically as $t^{-\frac{1}{2}}$ for the two dimensional case and as t^{-1} for the three dimensional tsunamis. Several important features of the solution are discussed in some detail. As an application of the general theory, some physically realistic ocean floor disturbances are included in this paper.

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1. INTRODUCTION.

A significant deformation of an ocean floor caused by an underwater earthquake, landslide or volcanic eruption produces a seismic wave (often called a tsunami) on the ocean surface. Once the tsunami is generated, it propagates into more and more shallow ocean, and its propagation is influenced by the depth of the ocean. Both the wave amplitude and energy increase significantly toward the shoreline. Since the energy of the tsunami is dissipated very slowly, the large amplitude ocean waves then strike the coastline, producing a serious threat to life, wealth and economic resources in coastal regions. Indeed, among the various natural hazards, the tsunamis are found to have catastrophic effects on near and distant coastal regions, coastal structure, marine vehicles and equipment.

Van Dorn (1) and Carrier (2) have made some informative surveys on the dynamics of tsunamis and discussed the qualitative and quantitative results obtained for various simplified modes of tsunamis. In order to indicate our interest in the problem of generation and propagation of tsunami in a running ocean, mention may be made of the two recent models of the ocean floor disturbances. Based upon the linear and nonlinear theory of wave motion in an ocean of uniform depth, Hammack (3) has made both theoretical and experimental investigation of tsunami generation due to some particular ocean floor disturbances. The main emphasis of his analysis is to determine the applicability of the linear theory in the generation region. Although this model does consider the nonlinearity, the surface displacement integrals related to ocean floor disturbances have not been evaluated so that no conclusion can be drawn about the principal features of the wave motions. Braddock et al (4) have considered the problems of tsunami generation due to a sea floor disturbance which is described by series of orthogonal functions. Using the standard techniques of integral transforms and stationary phase methods, they have presented the asymptotic solution for the free surface flows produced by the applied ocean floor disturbance. It has been shown that tsunami consists of a dispersive wave train preceded by a nondispersive wave front traveling as a long ocean wave. The relative order of magnitude of the wave train and the wave front is found to depend on the degree of symmetry or asymmetry of the ocean floor disturbances.

In almost all models considered in the literature including the two mentioned above, the dynamics of tsunamis was confined to shallow or deep oceans at rest. In this paper, a study is made of the generation and propagation of tsunamis in a shallow running ocean due to the action of an arbitrary ocean floor or ocean surface disturbance. An asymptotic analysis for both two and three dimensional tsunamis is presented to investigate the principal features of the free surface elevation. It is shown that the propagation of the tsunamis depends on the relative magnitude of the speed of the running ocean and the critical wave speed in the shallow ocean. When the basic speed, U, of the running ocean is less than the speed of the shallow ocean wave, both the two and the three dimensional free surface elevation represent the generation and propagation of surface waves which decay asymptotically as $t^{-\frac{1}{2}}$ for the two dimensional problem and as t^{-1} for the three dimensional tsunamis.

2. MATHEMATICAL FORMULATION OF GENERAL PROBLEM.

We consider an inviscid incompressible fluid of infinite horizontal extent which is bounded above by the free surface at $z = 0$ and bounded below by a solid bottom at $z = -h$. In its undisturbed state, the fluid flows with constant velocity. The wave motion is set up in the fluid by the combined action of a given bottom disturbance and the pressure distribution at the free surface. It is convenient to formulate the initial value problem in a coordinate frame with respect to which the applied pressure is at rest. We thus take the Cartesian coordinate system Oxyz such that the $x - y$ plane represents the undisturbed free surface with the origin located on it, the z - axis is directed vertically upward and the fluid moves in the Ox direction with uniform velocity U relative to this frame.

At time $t > 0$, the solid bottom of the ocean is subjected to move in a prescribed manner given by $z = -h + \zeta\ (\underline{r},\ t)$ such that $\zeta\ (\underline{r},\ t) \to 0$ as $r \to \infty$ when $\underline{r} = (x,\ y)$. In addition, the pressure is prescribed on the free surface of the liquid so that the free surface flow is generated in the shallow running ocean by the action of the surface pressure or the bottom disturbance.

As the flow is generated by the disturbances in the uniform stream, the disturbance velocity potential $\phi(x,y,z;t)$ satisfies the Laplace equation

$$\nabla^2\phi \equiv \phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \quad (x,\ y)\ \varepsilon\ (-\infty,\ \infty)\ -h \le z \le 0,\ t > 0. \tag{2.1}$$

With $\eta \equiv \eta(x,y;t)$ representing free surface displacement, the linearized kinematic boundary conditions on the free surface and the solid bottom are

$$\phi_z = \eta_t + U\eta_x \qquad \text{on } z = 0,\ t > 0, \tag{2.2}$$

$$\phi_z = \zeta_t + U\zeta_x \qquad \text{on } z = 0,\ t > 0, \tag{2.3}$$

In the absence of surface tension of the fluid, the dynamic condition on the free surface in the linearized form is given by

$$\phi_t + U\phi_x + g\eta = -\frac{1}{\rho}\, p(x,y;t) \qquad \text{on } z = 0,\ t > 0, \tag{2.4}$$

where ρ is the constant density of the fluid and $p(x,y;t)$ is the pressure prescribed on the face surface.

The initial conditions are

$$\phi(x,y,z;0) = \eta(x,y;0) = \zeta(x,y;0) = 0, \tag{2.5}$$

Further, it is assumed that ϕ, ζ and ζ are generalized functions (or distributions) of x and y in the sense of Lighthill (5) so that their Fourier transforms exist with respect to x and y.

3. SOLUTION OF THE INITIAL VALUE PROBLEM.

The above wave problem can readily be solved by using the Laplace transform with respect to t and the generalized Fourier transform with respect to x and y defined by

$$\tilde{\bar{f}}(\underline{k},z;s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\underline{k}\cdot\underline{r}}\, d\underline{r} \int_{0}^{\infty} e^{-st} f(\underline{r},z;t)dt, \tag{3.1}$$

where $\underline{k} \equiv (k,\ell)$ is the two-dimensional wave vector, the tilda and the bar denote the Laplace and the Fourier transforms respectively.

Application of (3.1) to the differential system (2.1) - (2.5) gives the solutions for the transform functions $\tilde{\bar{\phi}}(\underline{k},z;s)$ and $\tilde{\bar{\eta}}(\underline{k},s)$. Using the inverse Laplace and Fourier transformations, the surface displacement $\eta(x,y;t)$ is given by

$$\eta(x,y;t) = \frac{1}{4\pi i}\int_{-\infty}^{\infty}\int_{c-i\infty}^{c+i\infty} \tilde{\bar{\eta}}(\underline{k},s)\, \exp(i\underline{k}\cdot\underline{r} + st)d\underline{k}ds, \tag{3.2}$$

where

$$\tilde{\bar{\eta}}(\underline{k},s) \equiv \frac{\tilde{\bar{\zeta}}(\underline{k},s)\,(s+Uik)^2 \operatorname{sech}|\underline{k}|h - (g\rho)^{-1} b^2\, \tilde{\bar{p}}(\underline{k},s)}{(s+Uik)^2 + b^2}, \tag{3.3}$$

$\tilde{\bar{\zeta}}(\underline{k},s)$, $\tilde{\bar{p}}(\underline{k},s)$ are the Laplace and the Fourier transforms of $\zeta(\underline{r},t)$ and $p(\underline{r},t)$, and

$$b \equiv (g|\underline{k}|\tanh|\underline{k}|h)^{\frac{1}{2}}, \tag{3.4}$$

It is noted that in the absence of the surface pressure disturbance and the basic stream, the integral solution (3.2) with (3.3) reduces to that of Hammack (3) for the two dimensional case and to that of Braddock et al (4) for the three dimensional case.

The Laplace inversion of (3.3) can be carried out by means of a suitable complex contour integral combined with the theory of residues. The simple poles of the integral of (3.2) are at $s = ib_1, -ib_2$ where $b_1 = b - Uk$ and $b_2 = b + Uk$. These poles are on the imaginary axis of the complex s-plane and their residue contributions to $\eta(x,y;t)$ lead to the propagation of surface wave trains. The other singularities of (3.2), if any, related to some physically realistic bottom or pressure disturbances are all poles located at the left of the imaginary axis in the s-plane. Their residue contributions to the solution decay exponentially in time and so are in general insignificant.

The residue contributions only from the simple poles at $s = ib_1, -ib_2$ give the oscillatory surface elevation in the form

$$\eta(x,y;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\underline{k},-ib_2)\exp(i\underline{k}\cdot\underline{r} - ib_2t)d\underline{r} - \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\underline{k},ib_1)\exp(i\underline{k}\cdot\underline{r} + ib_1t)d\underline{k}, \tag{3.5}$$

where

$$A(\underline{k},s) = \frac{1}{2ib}[\tilde{\bar{\zeta}}(\underline{k},s)(s+Uik)^2 \operatorname{sech}|k|h + (g\rho)^{-1} b^2\tilde{\bar{p}}(\underline{k},s)], \tag{3.6}$$

It is evident that the terms $\exp[i(\underline{k}\cdot\underline{r} \pm tb_n)]$, $n = 1, 2$ represent the surface waves with the complex amplitudes $A(\underline{k}, \pm ib_n)$ respectively.

On the other hand, when $\tilde{\bar{\zeta}}(\underline{k},s)$ and $\tilde{\bar{p}}(k,s)$ have polar singularities on the imaginary axis in the s-plane, the Laplace inversion of (3.3) would be made up of these polar contributions including from ib_1 and $-ib_2$.

Of special interest are the following bottom and pressure disturbances:

$$\zeta(\underline{r},t) = \zeta_0\zeta(\underline{r})f(t)H(t), \quad p(\underline{r},t) = Pp(r)e^{i\omega t}H(t), \tag{3.7a,b}$$

where ζ_0, P are constants, H(t) is the Heaviside function of time t, $\zeta(\underline{r}) \equiv \zeta(x,y)$, and $p(\underline{r}) \equiv p(x,y)$ are functions of compact support in the x-y plane, f(t) is a suitable function of t and ω is the frequency of the applied pressure.

The surface displacement $\eta(x,y;t)$ related to (3.7a,b) is obtained from (3.2) in the form

$$\eta(x,y;t) = \frac{\zeta_0}{2\pi}\int_{-\infty}^{\infty} \frac{\bar{\zeta}(k)}{\cosh|\underline{k}|h}\,[f(t) - b\int_0^t e^{-iU\tau k}f(t-\tau)\sin b\tau\, d\tau]e^{i\underline{k}\cdot\underline{r}}d\underline{k}$$

$$+ \frac{P}{2\pi g\rho}\int_{-\infty}^{\infty} b\bar{p}(\underline{k})\,[(\frac{1}{\omega-b_1} - \frac{1}{\omega+b_2})e^{i\omega t} - (\frac{e^{ib_1t}}{\omega-b_1} - \frac{e^{-ib_2t}}{\omega+b_2})]e^{i\underline{k}\cdot\underline{r}}d\underline{k}, \tag{3.8}$$

In general, the exact evaluation of the integrals (3.5) and (3.8) is almost a formidable task and hence it is necessary to resort to asymptotic methods.

4. ASYMPTOTIC ANALYSIS FOR THE TWO DIMENSIONAL PROBLEM.

In the corresponding two dimensional wave problem, there is no y dependence. Hence, the free surface displacement $\eta(x,t)$ corresponding to (3.5) is given by

$$\eta(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k,-ib_2)\exp(ikx - ib_2t)dk - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k,ib_1)\exp(ikx + ib_1t)dk, \tag{4.1}$$

where k is the one dimensional wave number,

$$A(k,s) \equiv \frac{1}{2ib}[\bar{\zeta}(k,s)(s + Uik)^2 \operatorname{sech} kh + (g\rho)^{-1}b^2\,\bar{p}(k,s)], \tag{4.2}$$

and $$b \equiv (gk \tanh kh)^{\frac{1}{2}}, \tag{4.3}$$

In order to make an asymptotic evaluation of (4.1) for large t such that $|x| << Ut$, we shall follow the method of Debnath and Rosenblat (1969). Writing χ for kh, the stationary points for the integrals in (4.1) as $t \to \infty$ are approximately given by $c\mu'(\chi) = \mp U$ where $c = \sqrt{gh}$ and $\mu(\chi) = (\chi \tanh\chi)^{\frac{1}{2}}$. The necessary and sufficient condition for the existence of stationary points is $U \le c$. And if $U > c$, there are no stationary points of the integrals in (4.1).

From a graphical method similar to that of Debnath and Rosenblat (6), it is easy to locate the roots of the equations $c\mu'(\chi) = \pm U$ for $U \le c$. Hence it follows that the first and the second integrals in (4.1) have stationary points at $k = -k_2$, $(k_2 > 0)$ and $k = k_1 > 0$ respectively for $U \le c$. Invoking the standard formula for the stationary phase expansion and incorporating the existence condition for the stationary points through the Heaviside function, the asymptotic representation of (4.1) for large t is given by

$$\eta(x,t) \sim H(c-U)\Big[\frac{A(-k_2, -ib_2)}{|t\, b''(-k_2)|^{\frac{1}{2}}} \exp\{-i(k_2 x + tb_2^* + \frac{\pi}{4})\}$$

$$- \frac{A(k_1, ib_1^*)}{|t\, b''(k_1)|^{\frac{1}{2}}} \exp\{i(k_1 x + tb_1^* + \frac{\pi}{4})\}\Big] + 0(\frac{1}{t}), \tag{4.4}$$

where $b_2^* = b(-k_2) - Uk_2$ and $b_1^* = b(k_1) - Uk_1$. (4.5a,b)

This represents the generation and propagation of surface wave trains which decay asymptotically as $t^{-\frac{1}{2}}$.

It is also noted that result (4.4) includes the critical case $U = c$ in the sense that then $k_1 = k_2 = 0$ and the contribution to $\eta(x,t)$ is of the order t^{-1}. Thus the wave system given by (4.4) decays more slowly than the wave front at the origin.

For the case $U > c$, the integrals do not have any stationary points and consequently they decay like t^{-1} as $t \to \infty$. Physically, this means that the free surface elevation has not yet penetrated into the region $|x| << |U-c|t << Ut$.

In order to describe the ultimate wave system, we next evaluate the integral solution (3.8) for the two dimensional problem with $f(t) = 1 - e^{-\alpha t}$, $\alpha > 0$. It turns out that (3.8) has the form

$$\eta(x,t) = \frac{P}{g\rho\sqrt{2\pi}} (I_1 e^{i\omega t} - J_1) + \frac{\zeta_0}{\sqrt{2\pi}} (I_2 - I_3 - I_4 + J_2 + J_3), \tag{4.6}$$

where the integrals $I_n (n = 1, 2, 3, 4)$, $J_n (n = 1, 2, 3)$ are given by

$$I_1 = \int_{-\infty}^{\infty} b\, \bar{p}(k) \left(\frac{1}{\omega-b_1} - \frac{1}{\omega+b_2}\right) e^{ikx}\, dk, \tag{4.7}$$

$$J_1 = \int_{-\infty}^{\infty} b\, \bar{p}(k) \left(\frac{e^{ib_1 t}}{\omega-b_1} - \frac{e^{-ib_2 t}}{\omega+b_2}\right) e^{ikx}\, dk, \tag{4.8}$$

$$I_2 = (1 - e^{-\alpha t}) \int_{-\infty}^{\infty} \frac{\bar{\zeta}(k)}{\cosh kh} e^{ikx}\, dk, \tag{4.9}$$

$$I_3 = \frac{1}{2} \int_{-\infty}^{\infty} \frac{b\, \bar{\zeta}(k)}{\cosh kh} \left(\frac{1}{b_1} + \frac{1}{b_2}\right) e^{ikx}\, dk, \tag{4.10}$$

$$I_4 = \frac{e^{-\alpha t}}{2i} \int_{-\infty}^{\infty} \frac{b\, \bar{\zeta}(k)}{\cosh kh} \left(\frac{1}{\alpha + ib_1} - \frac{1}{\alpha - ib_2}\right) e^{ikx}\, dk, \tag{4.11}$$

$$J_2 = \frac{1}{2} \int_{-\infty}^{\infty} \frac{b\, \bar{\zeta}(k)}{\cosh kh} \left(\frac{e^{ib_1 t}}{b_1} + \frac{e^{-ib_2 t}}{b_2}\right) e^{ikx}\, dk, \tag{4.12}$$

$$J_3 = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{b\,\bar{\zeta}\,(k)}{\cosh\,kh}\,\left(\frac{e^{ib_1 t}}{\alpha + ib_1} - \frac{e^{-ib_2 t}}{\alpha - ib_2}\right) e^{ikx}\,dk, \tag{4.13}$$

It is convenient to write (4.6) in the form $\eta(x,t) = \eta_1(x,t) + \eta_2(x,t)$, (4.14) where η_1 consists of I_1 and J_1, and represents the free surface displacement due to the surface pressure distribution; η_2 is made up of $I_n(n = 2, 3, 4)$ and $J_n(n = 2, 3)$, and describes also the surface displacement originated entirely due to the bottom disturbance.

Integrals I_1 and J_1 are exactly identical with those obtained by Debnath and Rosenblat (6). A detailed asymptotic analysis of these integrals was presented in that paper. It may therefore be fair to avoid duplication, and to quote some important results without any further elaboration. It follows from paper (6) that the asymptotic solution $\eta_1(x,t)$ consists of the steady-state and the transient wave components. For large $|x|$ and t such that $Ut >> |x|$, the latter is of the order $t^{-\frac{1}{2}}$ for $c > U$, and decays asymptotically as $t \to \infty$. Thus the ultimate steady-state wave system is established in the limit $t \to \infty$ and can be written in terms of notations of (6) as

$$\eta_{1(st)}(x,t) \sim \frac{Pi}{\rho g}\sqrt{\frac{\pi}{2}}\; e^{i\omega t}[\phi_1(-s_1)e^{-s_1 x} + \psi_1(-s_2)\; e^{-s_2 x}$$
$$+ H(c - U)\;\phi_1(\sigma_2)\; e^{i\sigma_2 x}], \text{ when } x > 0, \tag{4.15}$$

and

$$\eta_{1(st)}(x,t) \sim \frac{Pi}{\rho g}\sqrt{\frac{\pi}{2}}\; H(c - U)\;\phi_1(\sigma_1)\; e^{i(\omega t+\sigma_1 x)}, \text{ when } x < 0, \tag{4.16}$$

where

$$\phi_1(\kappa) = \frac{b(\kappa)\;\bar{p}\;(\kappa)}{\left(\frac{db_1}{dk}\right)_{k=\kappa}} \quad \text{and} \quad \psi_1(\kappa) = \frac{b(\kappa)\;\bar{p}\;(\kappa)}{\left(\frac{db_2}{dk}\right)_{k=\kappa}} \tag{4.17a,b}$$

The steady-state solution represents the propagation of either two or four surface waves according as $U > c$ or $U < c$.

At the critical speed, $U = c$, both the steady-state and the transient solutions asymptotically tend to infinity. Indeed, the former represents a wave whose amplitude increases like x while the latter is also a wave whose amplitude is of the order $t^{\frac{1}{2}}$ as $t \to \infty$. This clearly suggests that the linearized theory based on small amplitude disturbances fails to provide with a physically sensible solution at the critical speed. Naturally, it would be necessary to include nonlinear terms in the original formulation of the problem in order to achieve a physically reasonable solution.

We next turn our attention to the asymptotic evaluation of I_n and $J_n (n = 2, 3, 4)$. These integrals have infinitely many pure imaginary poles at $(n + \frac{1}{2}) \frac{\pi i}{h}$, $n = 0, \pm 1, \pm 2, \ldots$; and can readily be evaluated by using the residue theory over a suitable contour. The residue contributions from these poles are insignificant as t or $|x| \to \infty$. The integral I_2 does not have any significant contribution to the free surface displacement.

From a graphical representation of $\mu(\chi) = \pm \frac{U}{c} \chi$ similar to that in (6), it follows that when $U < c$, integral I_3 has two real poles at $k = \pm k_3$ (say), and when $U \geq c$, it has no real poles. Using the formula (24) of Debnath and Rosenblat (6), the residue contribution from these poles, as $|x| \to \infty$, is given by

$$I_3 \sim \frac{\pi i}{2} \operatorname{sgn} x \, H(c - U) [\phi_2(k_3) \, e^{ik_3 x} + \psi_2(-k_3) \, e^{-ik_3 x}], \tag{4.18}$$

where

$$\phi_2(\kappa) = \frac{b(\kappa) \, \bar{\zeta}(\kappa)}{\cosh \kappa h \; b_1'(\kappa)}, \qquad \psi_2(\kappa) = \frac{b(\kappa) \, \bar{\zeta}(\kappa)}{\cosh \kappa h \; b_2'(\kappa)}, \tag{4.19a,b}$$

The transient integral J_2 has the same poles as those of I_3, and for $t \to \infty$, it has stationary points when $\mu'(\chi) = \pm \frac{U}{c}$. The contribution to J_2 from its stationary points and the polar contribution to J_2 can be evaluated asymptotically for large t by the same method employed in paper (6). The asymptotic value of J_2 as $t \to \infty$ arising from its poles are given by

$$J_{2(polar)} \sim -\frac{\pi i}{2} H(c - U)[\phi_2(k_3)\, e^{ik_3x} + \psi_2(-k_3)\, e^{-ik_3x}], \tag{4.20}$$

The transient part of J_2 is made up of the contribution from the stationary points and is given by

$$J_2 \sim \frac{1}{2} H(c-U)[\frac{(t|b''(-k_2)|)^{-\frac{1}{2}}}{\cosh k_2h} \bar{\zeta}(-k_2)\, b_2^* \exp\{-i(k_2x + tb_2^* + \frac{\pi}{4})\}$$

$$- \frac{(t|b''(k_1)|)^{-\frac{1}{2}}}{\cosh k_1h} \bar{\zeta}(k_1)b_1^* \exp\{i(k_1x + tb_1^* + \frac{\pi}{4})\}] + 0(\frac{1}{t}), \tag{4.21}$$

At the critical speed, $U = c$, J_2 has a stationary point at $k = 0$ and has no polar singularities. Since $b''(0) = 0$, the asymptotic expansion (4.18) is no longer valid at the origin. Near the wave front at $k = 0$, the asymptotic value of J_2 can be found from Copson (9) as

$$J_2 \sim \frac{1}{2} \frac{\Gamma(\frac{1}{3})}{\sqrt{3}} A(0)(6t|b'''(0)|)^{-\frac{1}{3}} + 0(t^{-\frac{2}{3}}), \tag{4.22}$$

where $A(0) = \lim_{k\to 0} A(k)$, $A(k) = \frac{b(k)\ \bar{\zeta}(k)}{\cosh kh\ b_1(k)}$ and $\Gamma(x)$ is the Gamma function.

It should be noted here that the wave front decays more slowly than the main transient wave system described by (4.2).

Integrals I_4 and J_3 have infinite sets of purely imaginary poles at $\pm i\beta_n$ and $\pm i\beta_n'$ where β_n and β_n' satisfy the equations

$$c(\beta\tan\beta)^{\frac{1}{2}} \mp U\beta - h\alpha = 0, \quad c(\beta'\tan\beta')^{\frac{1}{2}} \pm U\beta' - h\alpha = 0, \tag{4.23a,b}$$

Evidently, as $t \to \infty$, these integrals do not have any significant contribution from these imaginary poles.

On the other hand, J_3 has the same stationary points as those of J_2 and its contribution from the stationary points for large t can be written in the form

$$J_3 \sim O(t^{-\frac{1}{2}}) \text{ or } O(t^{-\frac{1}{3}}), \quad \text{according as } c > U \text{ or } c = U. \tag{4.24a,b}$$

Thus the transient component of $\eta_2(x,t)$ decays asymptotically as $t \to \infty$ and the ultimate steady-state is reached which takes the asymptotic form:

$$\eta_{2(st)}(x,t) \sim -i\zeta_0\sqrt{\frac{\pi}{2}}\, H(c-U)\,[\phi_2(k_3)e^{ik_3x} + \psi_2(-k_3)e^{-ik_3x}], \text{ when } x > 0 \tag{4.25a}$$

$$\sim 0 \quad \text{when } x < 0. \tag{4.25b}$$

Hence all the wave integrals involved in the free surface displacement $\eta(x,t)$ given by (4.6) have been evaluated asymptotically for large $|x|$ or t. It follows from the above analysis that the transient component of $\eta(x,t)$ decays asymptotically as $t \to \infty$. And the ultimate steady-state wave system is attained and consists of $\eta_{1(st)}$ and $\eta_{2(st)}$ given by (4.15), (4.16) and (4.25a,b).

5. SOME PARTICULAR DISTURBANCES FOR TSUNAMI GENERATION.

It is of interest to mention some physically realistic form of disturbances for the generation of tsunami:

(a) $p(x) = \delta(x)$, (b) $p(x) = \frac{1}{\varepsilon\sqrt{\pi}} \exp(-\frac{x^2}{\varepsilon^2})$

(c) $p(x) = H(a-|x|)$, (d) $\zeta(x) = \delta(x)$, $f(t) = e^{i\omega^* t}$

(e) $\zeta(x) = H(a-|x|)$, $f(t) = \frac{1}{2}(1-\cos\frac{\pi t}{t})H(T-t) + H(t-T)$

(f) $\zeta(x) = \frac{1}{\varepsilon\sqrt{\pi}} \exp(-\frac{x^2}{\varepsilon^2})$, $f(t) = (1 - e^{-\alpha t})$, $\alpha > 0$;

where $\delta(x)$ is the Dirac delta function.

6. THREE DIMENSIONAL TSUNAMIS.

The integral representation for the free surface displacement $\eta(x,y;t)$ due to an arbitrary ocean floor disturbance only is given by

$$\eta(x,y;t) = \frac{1}{4\pi i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{b(\underline{k})}{\cosh|\underline{k}|h}[\bar{\zeta}(\underline{k}, ib_1) \exp(i\underline{k}\cdot\underline{r} + ib_1 t)$$

$$- \bar{\zeta}(\underline{k}, -ib_2) \exp(i\underline{k}\cdot\underline{r} - ib_2 t)]d\underline{k}, \tag{6.1}$$

Invoking the polar coordinate transformation $(x,y) = (r \cos\theta, r \sin\theta)$ and $(k,\ell) = (\kappa \cos\psi, \kappa \sin\psi)$, integral (6.1) takes the form

$$\eta(x,y;t) = \frac{c}{4\pi i h^2}\int_0^{\infty}\int_0^{2\pi} \frac{\lambda\mu(\lambda)}{\cosh\lambda} [\bar{\zeta}_+ (\lambda,\psi) \exp\{itf^+(\lambda,\psi)\}$$

$$-\bar{\zeta}_-(\lambda,\psi) \exp\{it\, f^-(\lambda,\psi)\}]d\lambda\, d\psi, \tag{6.2}$$

where

$$\bar{\zeta}_\pm(\lambda,\psi) \equiv \bar{\zeta}(\frac{\lambda}{h} \cos\psi, \frac{\lambda}{h} \sin\psi; \qquad \pm \frac{i}{h}[c\mu(\lambda) \mp U\lambda \cos\psi]), \tag{6.3a,b}$$

$$f^\pm(\lambda\ \psi) \equiv \frac{1}{h}[\frac{\lambda r}{t} \cos(\psi-\theta) \pm c\mu(\lambda) - U\lambda \cos\psi], \tag{6.4a,b}$$

and

$$\mu(\lambda) = (\lambda \tanh\lambda)^{\frac{1}{2}}, \qquad \lambda = \kappa h, \tag{6.5a,b}$$

The exact evaluation of (6.2) is almost a formidable task. We then apply the method of stationary phase twice to the integral (6.2). The points of stationary phase related to the integrals of (6.2) are the solution of the equations

$$\frac{\partial f^\pm}{\partial\lambda} = 0 \qquad \text{and} \qquad \frac{\partial f^\pm}{\partial\psi} = 0, \tag{6.6a,b}$$

The main interest is in finding the asymptotic expansion of (6.2) for large r and t such that $0 < r << Ut$. Consequently, the approximate form of (6.6a,b) as $t \to \infty$ is sufficient to determine the points of stationary phase involved in (6.2). Thus (6.6b) gives the stationary points approximately at $\psi = 0, \pi$ and

2π. With these values of ψ, and $r = x$, (6.6a) reduces to the corresponding equations for the two dimensional problem. Hence the existence condition for the stationary points of (6.2) is the same as that discussed in Section 4. Evidently, for $U \leq c$, the first equation in (6.6a) gives one non-negative stationary point at $\lambda = \lambda_1$ when $\psi = 0$ or 2π; and the second equation in (6.6a) has also a non-negative stationary point at $\lambda = \lambda_1$ when $\psi = \pi$. Thus the first and the second integrals of (6.2) have stationary points at $(\lambda_1, 0)$ and (λ_1, π) respectively.

Application of the stationary phase approximation for the double integral in (6.2) as $t \to \infty$ (Debnath, 7) yields

$$\eta \sim \frac{cH(c-H)}{4\pi i\, h^3} \left\{ \frac{2\pi\lambda_1\mu(\lambda_1)}{t \cosh \lambda_1} \left[\frac{\varepsilon_1^+ \varepsilon_2^+ \bar{\zeta}_+ (\lambda_1, 0)}{|D+|^{\frac{1}{2}}} \exp\{itf^+(\lambda_1, 0)\} \right.\right.$$

$$\left.\left. - \frac{\varepsilon_1^- \varepsilon_2^- \bar{\zeta}_- (\lambda_1, \pi)}{|D-|^{\frac{1}{2}}} \exp\{itf^-(\lambda_1,\pi)\} \right] \right\} + 0(\frac{1}{t^2}), \tag{6.7}$$

where

$$\varepsilon_1^+ = \exp[\frac{i\pi}{4} \operatorname{sgn}\{f_{\lambda\lambda}^+(\lambda_1, 0)\}], \quad \varepsilon_2^+ = \exp[\frac{i\pi}{4} \operatorname{sgn}\{f_{\psi\psi}^+(\lambda_1, 0)\}]$$

$$\varepsilon_1^- = \exp[\frac{i\pi}{4} \operatorname{sgn}\{f_{\lambda\lambda}^-(\lambda_1, \pi)\}], \quad \varepsilon_2^- = \exp[\frac{i\pi}{4} \operatorname{sgn}\{f_{\psi\psi}^-(\lambda_1, \pi)\}]$$

$$D_+ \equiv f_{\lambda\lambda}^+(\lambda_1, 0)\, f_{\psi\psi}^+(\lambda_1, 0) - \{f_{\lambda\psi}^+(\lambda_1, 0)\}^2, \tag{6.8}$$

and

$$D_- \equiv f_{\lambda\lambda}^-(\lambda_1,\pi)\, f_{\lambda\lambda}^-(\lambda_1,\pi) - \{f_{\lambda\psi}^-(\lambda_1,\pi)\}^2, \tag{6.9}$$

It is evident that the asymptotic solution (6.7) for the free surface elevation represents the generation and propagation of the sinusoidal surface wave trains of complex amplitude which decays as t^{-1}.

At the critical case, $U = c$, the first and the second integrals in (6.2) have stationary points at $(0, 0)$ and $(0, \pi)$ respectively. It turns out that the solution (6.7) includes this critical case and the asymptotic contribution to the free surface elevation is of the order t^{-2}. Thus the wave front decays more rapidly than the main wave system described by (6.7).

Finally, in the supercritical case, $U > c$, there are no points where the phase functions of the integrals in (6.2) are stationary. In the absence of the stationary points, the solution decays asymptotically at least like t^{-1}.

7. MODELLING THE OCEAN FLOOR DISTURBANCES.

It is important for the study of tsunamis to model the ocean floor disturbances responsible for tsunami generation. As an application of the general theory presented in the previous section, we shall mention some physically realistic form of $\zeta(\underline{r})$ and $f(t)$ involved in (3.7a).

(a) $$\zeta(\underline{r}) = \exp(-\alpha x^2-\beta y^2), \quad \bar{\zeta}(\underline{k}) = \frac{\exp[-(\frac{k^2}{4\alpha}) + \frac{\ell^2}{4\beta})]}{2\sqrt{\alpha\beta}}, \quad \alpha,\beta > 0$$

(b) $$\zeta(\underline{r}) = \delta(x)\,\delta(y), \quad \bar{\zeta}(\underline{k}) = \frac{1}{2\pi}$$

(c) $$\zeta(\underline{r}) = \frac{1}{4ab} H(|x|-a)\, H(|y|-b), \quad \bar{\zeta}(\underline{k}) = \frac{1}{2\pi a\ell}\frac{\sin ka}{k}\frac{\sin \ell b}{\ell}$$

(d) $$\zeta(\underline{r}) = \frac{1}{a\ell} J_0[b(a^2-x^2-y^2)^{\frac{1}{2}}]\, H(a^2-x^2-y^2), \quad b > 0$$

$$\bar{\zeta}(\underline{k}) = \frac{J_1[a(b^2+k^2+\ell^2)^{\frac{1}{2}}]}{(b^2+k^2)^{\frac{1}{2}}\,(b^2+k^2+\ell^2)^{\frac{1}{2}}},$$ [Erdélyi, 8, 1.13 (47) and 1.7 (37)]

(e) $$\zeta(\underline{r}) = \frac{\nu}{\pi ab}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\nu-1} H\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

$$\bar{\zeta}(\underline{k}) = \frac{2^{\nu-1}\nu\Gamma(\nu)\, J_\nu[(k^2a^2+\ell^2b^2)^{\frac{1}{2}}]}{\pi(k^2a^2+\ell^2b^2)^{\nu/2}},$$ [Erdélyi, 3, 1.3 (8) and 1.13 (50)]

(f) $$\zeta(\underline{r}) = \frac{\nu xy}{\pi a^3 b^3}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\nu-1} H\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right),$$

$$\bar{\zeta}(\underline{k}) = \frac{2^{\nu-1}\nu\Gamma(\nu)\, k\ell J_{\nu+2}[(k^2a^2 + \ell^2b^2)^{\frac{1}{2}}]}{\pi(k^2a^2 + \ell^2b^2)^{1+\nu/2}},$$ [Erdélyi, 8, 2.3 (9) and 2.13 (51)]

and $f(t) = e^{-\omega t}(\omega > 0)$.

We note that (a)-(e) represent symmetric bottom disturbances and (f) is an asymmetric disturbance. Also, the disturbance (a) is distributed over a finite region, (d) and (e)-(f) are confined to an exactly circular region and elliptic region respectively.

Finally, it may be added that in addition to the above ocean floor disturbances, the present analysis incorporates other forms of physically realistic disturbances for the generation of tsunamis.

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