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A CONVOLUTION PRODUCT OF $(2j)$th DERIVATIVE OF DIRAC'S DELTA IN r AND MULTIPLICATIVE DISTRIBUTIONAL PRODUCT BETWEEN r^{-k} AND $\nabla(\triangle^j\delta)$

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The purpose of this paper is to obtain a relation between the distribution $\delta^{(2j)}(r)$ and the operator $\triangle^j\delta$ and to give a sense to the convolution distributional product $\delta^{(2j)}(r)*\delta^{(2s)}(r)$ and the multiplicative distributional products $r^{-k}\cdot\nabla(\triangle^j\delta)$ and $(r-c)^{-k}\cdot\nabla(\triangle^j\delta)$.

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1. Introduction. Let $x=(x_1,x_2,\dots,x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$.

We call $\varphi(x)$ the C^∞-functions with compact support defined from $\mathbb{R}^n$ to $\mathbb{R}$. Let

$$r^2=x_1^2+x_2^2+\cdots+x_n^2 \tag{1.1}$$

and consider the functional r^λ defined by

$$(r^\lambda,\varphi)=\int_{\mathbb{R}^n} r^\lambda\varphi(x)\,dx \tag{1.2}$$

(see [5, page 71]), where λ is a complex number and $dx=dx_1dx_2\cdots dx_n$.

For $\operatorname{Re}(\lambda)>-n$, this integral converges and is an analytic function of λ. Analytic continuation to $\operatorname{Re}(\lambda)\le -n$ can be used to extend the definition of (r^λ,φ).

Calling Ω_n to the hypersurface area of the unit sphere imbedded in the n-Euclidean space, we find in [5, page 71] that

$$(r^\lambda,\varphi)=\Omega_n\int_0^\infty r^{\lambda+n-1}S_\varphi(r)\,dr, \tag{1.3}$$

where

$$S_\varphi(r)=\frac{1}{\Omega_n}\int_\Omega \varphi\,dw \tag{1.4}$$

and dw is the hypersurface element of the unit sphere.

$S_\varphi(r)$ is the mean value of $\varphi(x)$ on the sphere of radius r (cf. [5, page 71]).

The functional r^λ [5, pages 72–73] has a simple pole at

$$\lambda = -n-2j, \quad j = 0,1,\ldots, \tag{1.5}$$

and from [5, page 99], the Laurent series expansion of r^λ in a neighbourhood of $\lambda = -n-2j$, $j = 0,1,2,\ldots$, is

$$\begin{aligned} r^\lambda = {} & \frac{\Omega_n}{(2j)!}\delta^{(2j)}(r)\frac{1}{\lambda+n+2j} \\ & + \Omega_n r^{-2j-n} + \Omega_n(\lambda+n+2j)r^{-2j-n}\ln(r) + \cdots. \end{aligned} \tag{1.6}$$

In (1.6), r^{-2j-n} is not the value of the functional r^λ at $\lambda = -n-2j$ (in fact, it has a pole at his point) but is the value of the regular part of the Laurent expansion of r^λ at this point.

From [6, page 366, formula (3.4)], we know that the neutrix product of r^{-k} and $\nabla\delta$ on $\mathbb{R}^m$ exists and, furthermore,

$$r^{-2k}\circ\nabla\delta = -\frac{1}{2^{k+1}(k+1)!(m+2)\cdots(m+2k)}\sum_{i=0}^{m}\left(x_i\triangle^{k+1}\delta\right), \tag{1.7}$$

$$r^{1-2k}\circ\nabla\delta = 0, \tag{1.8}$$

where k is a positive integer, m is the dimension of the space, $\triangle^j$ is the iterated Laplacian operator defined by (1.10), and ∇ is the operator defined by

$$\nabla = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \cdots + \frac{\partial}{\partial x_n} = \sum_{i=1}^{n}\frac{\partial}{\partial x_i}. \tag{1.9}$$

In (1.7) and (1.8), by the symbol $\circ$ we mean "neutrix product" which is defined by Li in [6, page 363, Definition 1.4, formula (1.11)].

The purpose of this paper is to obtain a relation between the distribution $\delta^{(2j)}(r)$ and the operator $\triangle^j\delta$ and to give a sense to convolution distributional product $\delta^{(2j)}(r)*\delta^{(2s)}(r)$ and the multiplicative distributional products $r^{-k}\cdot\nabla(\triangle^j\delta)$ and $(r-c)^{-k}\cdot\nabla(\triangle^j\delta)$ which are showed in Sections 2, 3.1, 3.2, and 3.3. Here, $\triangle^j$ is defined by

$$\triangle^j = \left\{\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\right\}^j, \tag{1.10}$$

and ∇ is the operator defined by (1.9).

We observed that relation (2.3) cannot be deduced from the formula

$$\delta^{(n+2j-1)}(r) = a_{j,n} \triangle^j \delta \tag{1.11}$$

which appear in [1], where

$$a_{j,n} = \frac{2^n \pi^{(n-1)/2}(-1)^{n+2j-1}\Gamma(n/2+j+1/2)}{j!} \tag{1.12}$$

with n dimension of the space.

Our formulae (3.7) and (3.18) result in a generalization of the neutrix product (1.7) and (1.8), respectively, due to Li (cf. [6, page 366, formula 3.4]).

To obtain our results, we need the following formulae:

$$(\delta^{(k)}(r-c), \varphi) = (-1)^k \Omega_n \left[\frac{\partial^k}{\partial r^k}(r^{n-1} S_\varphi(r)) \right]_{r=c} \tag{1.13}$$

(see [3, page 58, formula (II, 2, 5)]), where

$$(\delta^{(k)}(r-c), \varphi) = \int \delta^{(k)}(r-c)\varphi\, dx = \frac{(-1)^k}{c^{n-1}} \int_{O_c} \frac{\partial^k}{\partial r^k}(\varphi r^{n-1})\, dO_c \tag{1.14}$$

(see [5, page 231, formula (10)]), O_c is the sphere $r-c=0$, and dO_c is the Euclidean element of area of it;

$$\operatorname{Res}_{\lambda=-n-2j}(r^\lambda, \varphi) = \frac{\Omega_n}{2^j j! n(n+2)\cdots(n+2j-2)} (\triangle^j \delta, \varphi) \tag{1.15}$$

(see [5, pages 72–73]), where

$$\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \tag{1.16}$$

$$\Gamma(z+k) = z(z+1)\cdots(z+k-1)\Gamma(z) \tag{1.17}$$

(see [4, page 3, formula (2)])

$$\Gamma(z)(1-z) = \frac{\pi}{\operatorname{sen}(z\pi)} \tag{1.18}$$

(see [4, page 3, formula (6)])

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \tag{1.19}$$

(see [4, page 5, formula (5)]), and

$$\operatorname{Res}_{\mu=-k,\ k=1,2,\ldots}(x_+^\mu, \varphi) = \frac{\varphi^{(k-1)}(0)}{(k-1)!} \tag{1.20}$$

(see [5, page 49]), where x_+^μ is the functional defined by

$$(x_+^\mu, \varphi) = \int_0^\infty x^\mu \varphi(x)\,dx \tag{1.21}$$

(see [5, page 48]), which is analytic for $\operatorname{Re}(\mu) > -1$ and can be analytically continued to the entire μ plane except for the point $\mu = -1, -2, \ldots$ where it has simple poles.

2. The relation between the distribution $\delta^{(2j)}(r)$ and the operator $\triangle^j\delta$. In this section, we want to obtain a formula that relates the distribution $\delta^{(2j)}(r)$ to the operator $\triangle^j\delta$.

From (1.12) and considering formula (1.13), the residue of (r^λ, φ) at $\lambda = -n-2j$ for nonnegative integer j is given by

$$\operatorname{Res}_{\lambda=-n-2j}(r^\lambda, \varphi) = \frac{\Omega_n \Gamma(n/2)}{2^{2j} j! \Gamma(n/2+j)} (\triangle^j \delta, \varphi), \tag{2.1}$$

where $\triangle^j$ is defined by (1.10) and Ω_n by (1.16), with n the dimension of the space and $j = 0, 1, 2, \ldots$.

From [5, page 72], S_φ is an even function of the simple variable r in K, where K is the space of infinitely differentiable functions with bounded support. Then, the $S_\varphi(r)$, where integral (1.3) represents the application of $\Omega_n x_+^\mu$ (with $\mu = \lambda + n - 1$) to x_+^μ, is defined by (1.6).

Using the Laurent series expansion of r^λ in a neighbourhood of $\lambda = -n-2j$, $j = 0, 1, 2, \ldots$, from (1.6), we have

$$\delta^{(2j)}(r) = \frac{(2j)!}{\Omega_n} \lim_{\lambda \to -n-2j} (\lambda + n + 2j) r^\lambda. \tag{2.2}$$

From (2.2) and using (2.1), we obtain the following formula:

$$\begin{aligned} \delta^{(2j)}(r) &= \frac{(2j)!}{\Omega_n} \lim_{\lambda \to -n-2j} (\lambda + n + 2j) r^\lambda \\ &= \frac{(2j)!}{\Omega_n} \operatorname{Res}_{\lambda \to -n-2j} r^\lambda = \frac{(2j)!\Gamma(n/2)}{2^{2j} j! \Gamma(n/2+j)} \triangle^j \delta. \end{aligned} \tag{2.3}$$

Using (1.17), formula (2.3) can be rewritten in the following form:

$$\delta^{(2j)}(r) = \frac{(2j)!}{j!} \frac{1}{2^j j! n(n+2)\cdots(n+2j-2)} \triangle^j \delta. \tag{2.4}$$

3. Applications of the basic formula (2.3). In this section, we want to give a sense to the convolution distributional product of the form $\delta^{(2j)}(r)*\delta^{(2s)}(r)$ and the distributional products $r^{-k}\cdot\nabla(\triangle^j\delta)$ and $(r-c)^{-k}\cdot\nabla(\triangle^j\delta)$.

3.1. The convolution distributional product of the form $\delta^{(2j)}(r)*\delta^{(2s)}(r)$. In this section, we designate $*$ the convolution.

We know from (2.3) that the following formula is true:

$$\delta^{(2j)}(r)=\frac{(2j)!\Gamma(n/2)}{2^{2j}j!\Gamma(n/2+j)}\triangle^j\delta. \tag{3.1}$$

From (3.1), $\delta^{(2j)}(r)$ is a finite linear combination of δ and its derivatives, in consequence, we conclude that $\delta^{(2j)}(r)$ is a distribution of the class O'_c, where O'_c [7, page 244] is the space of rapidly decreasing distributions. Therefore, using the formula

$$\triangle^t\delta*\triangle^s\delta=\triangle^{t+s}\delta \tag{3.2}$$

[2, page 75, formula (26)], where $\triangle^t$ is the iterated Laplacian operator defined by (1.10), we obtain the following formula:

$$\delta^{(2j)}(r)*\delta^{(2s)}(r)=b_{j,s,n}\delta^{(2(j+s))}(r), \tag{3.3}$$

where

$$b_{j,s,n}=\frac{(j+s)!(2s)!(2j)!\Gamma(n/2)\Gamma(n/2+j+s)}{j!(2(j+s))!s!\Gamma(n/2+s)\Gamma(n/2+j)}. \tag{3.4}$$

In particular, letting $j=s=0$ in (3.3), we have

$$\delta(r)*\delta(r)=\delta(r), \tag{3.5}$$

where $r=\sqrt[2]{x_1^2+x_2^2+\cdots+x_n^2}$.

3.2. The multiplicative distributional product of $r^{-k}\cdot\nabla(\triangle^j\delta)$. To give a sense to the multiplicative distributional product of

$$r^{-k}\cdot\nabla(\triangle^j\delta), \tag{3.6}$$

we must study the cases $r^{-2k}\cdot\nabla(\triangle^j\delta)$ and $r^{1-2k}\cdot\nabla(\triangle^j\delta)$ where ∇ is the operator defined by (1.9) and $\triangle^j$ is the iterated Laplacian operator defined by (1.10).

Our formulae (3.7) and (3.18) result in a generalization of the neutrix product (1.7) and (1.8), respectively, due to Li (cf. [6, page 366, formula (3.4)]).

Theorem 3.1. *Let k be a positive integer and let j be a nonnegative integer, then the formula*

$$r^{-2k}\cdot\nabla(\triangle^j\delta)=\frac{-(n+2j)(2j)!}{(k+j+1)!2^{k+j+1}n(n+2)\cdots(n+2(k+j))}\left(\sum_{i=1}^{n}x_i\right)\triangle^{k+j+1}\delta \tag{3.7}$$

is valid if $k\neq n/2,n/2+1,\dots,n/2+s$, $s=0,1,\dots$ where ∇ is the operator defined by (1.9) and $\triangle^j$ is defined by (1.10).

Proof. Using formula (2.4), we have

$$\begin{aligned}r^{-2k}\cdot\nabla(\triangle^j\delta)&=r^{-2k}\cdot\nabla\left(\frac{2^jn(n+2)\cdots(n+2j-2)j!}{(2j)!}\delta^{(2j)}(r)\right)\\&=\frac{2^jn(n+2)\cdots(n+2j-2)j!}{(2j)!}r^{-2k}\cdot\nabla\delta^{(2j)}(r)\end{aligned} \tag{3.8}$$

if $k\neq n/2,n/2+1,\dots,n/2+s$, $s=0,1,\dots$.

Now, using the properties

$$\frac{\partial}{\partial x_j}\delta^{(k)}(P)=\frac{\partial P}{\partial x_j}\delta^{(k+1)}(P) \tag{3.9}$$

(see [5, page 232]) for

$$P=P(x_1,x_2,\dots,x_n)=r=\sqrt[2]{x_1^2+x_2^2+\cdots+x_n^2} \tag{3.10}$$

and using formula (1.9), we have

$$\nabla\delta^{(2j)}(r)=\sum_{i=1}^{n}\delta^{(2j+1)}(r)\frac{x_i}{r}. \tag{3.11}$$

From (3.8) and (3.11), we have

$$r^{-2k}\cdot\nabla\delta^{(2j)}(r)=\sum_{i=1}^{n}x_i\left(r^{-2k-1}\delta^{(2j+1)}(r)\right). \tag{3.12}$$

On the other hand, using formula (2.2), we have

$$\begin{aligned}r^{-2k-1}\cdot\delta^{(2j+1)}(r)&=r^{-2k-1}\cdot\frac{\partial}{\partial r}\delta^{(2j)}(r)\\&=\frac{(2j)!(-n-2j)}{(2(k+j+1))!}\delta^{(2(k+j+1))}(r).\end{aligned} \tag{3.13}$$

From (3.13) and using formula (2.4), we have

$$\begin{aligned}&r^{-2k-1}\cdot\delta^{(2j+1)}(r)\\&\quad=\frac{(2j)!(-1)(n+2j)}{(k+j+1)!2^{k+j+1}n(n+2)\cdots(n+2(k+j+1)-2)}\triangle^{k+j+1}\delta.\end{aligned} \tag{3.14}$$

Therefore, from (3.13) and using (3.14), we obtain

$$\begin{aligned} r^{-2k}\cdot\nabla(\triangle^j\delta) &= \left(\sum_{i=1}^n x_i\right) r^{-2k-1}\cdot\delta^{(2j+1)}(r) \\ &= \frac{-(n+2j)(2j)!}{(k+j+1)!2^{k+j+1}n(n+2)\cdots(n+2(k+j))} \\ &\quad\times\left(\sum_{i=1}^n x_i\right)\triangle^{k+j+1}\delta. \end{aligned} \tag{3.15}$$

if $k \neq n/2, n/2+1, \dots, n/2+s$, $s = 0,1,\dots$.

Formula (3.15) coincides with formula (3.7). Theorem 3.1 and formula (3.7) generalize the neutrix product $r^{-2k}\circ\nabla\delta$ given by Li [6, page 366, Theorem 3.4, formula (3.4)].

In fact, letting $j = 0$ in (3.7) and using that

$$\triangle^0\delta = \delta, \tag{3.16}$$

we have

$$r^{-2k}\cdot\nabla\delta = -\frac{1}{(k+1)!2^{k+1}(n+2)\cdots(n+2k)}\left(\sum_{i=1}^n x_i\right)\triangle^{k+1}\delta. \tag{3.17}$$

Formula (3.17) coincides with formula (1.7). □

Theorem 3.2. *Let k be a positive integer and let j be a nonnegative integer, then the formula*

$$r^{1-2k}\cdot\nabla(\triangle^j\delta) = 0 \tag{3.18}$$

is valid if $k \neq n/2, n/2+1/2, \dots, n/2+s+1/2$, $s = 0,1,\dots$ where ∇ is the operator defined by (1.9) and $\triangle^j$ is defined by (1.10).

Proof. Using formulae (2.4), (3.9), and (3.11), we have

$$\begin{aligned} r^{1-2k}\cdot\nabla(\triangle^j\delta) &= r^{1-2k}\cdot\nabla\left(\frac{2^j j! n(n+2)\cdots(n+2j-1)}{(2j)!}\delta^{(2j)}(r)\right) \\ &= \frac{2^j j! n(n+2)\cdots(n+2j-1)}{(2j)!}\left(\sum_{i=1}^n x_i\right) r^{-2k}\cdot\delta^{(2j+1)}(r) \end{aligned} \tag{3.19}$$

if $k \neq n/2, n/2+1/2, \dots, n/2+s+1/2$, $s = 0,1,\dots$.

On the other hand, using formula (2.2) and the properties

$$\Gamma(\beta+1) = \beta\Gamma(\beta) \quad (\text{see (1.17)}), \tag{3.20}$$

we have

$$\begin{aligned} r^{-2k}\cdot\delta^{(2j+1)}(r) &= r^{-2k}\cdot\frac{\partial}{\partial r}\delta^{(2j)}(r) \\ &= -\frac{(n+2j)(2j)!}{\Omega_n}\lim_{\beta\to 0}\beta r^{\beta-n-2j-2k-1} \\ &= -\frac{(n+2j)(2j)!}{\Omega_n}\lim_{\beta\to 0}\frac{\Gamma(\beta+1)}{\Gamma(\beta)}r^{\beta-n-2j-2k-1}. \end{aligned} \tag{3.21}$$

Now, using that r^{λ} is regular at the points $\lambda=-n-2(j-k)-1$, $j=0,1,\dots$, $k=1,2,\dots$, and the properties

$$\lim_{\beta\to 0}\frac{1}{\Gamma(\beta)}=0 \tag{3.22}$$

(see (1.17)), we have

$$r^{-2k}\cdot\delta^{(2j+1)}(r)=0. \tag{3.23}$$

From (3.19) and using (3.23), we obtain

$$r^{1-2k}\cdot\nabla(\triangle^j\delta)=0 \tag{3.24}$$

if $k\neq n/2,n/2+1/2,\dots,n/2+s+1/2$, $s=0,1,\dots$.

Formula (3.24) coincides with formula (3.18). □

Theorem 3.2 and formula (3.15) generalized the Neutrix Product $r^{1-2k}.\nabla\delta$ given by Li [6, page 366, formula (3.4), Theorem 3.1]. In fact, letting $j=0$ in (3.18) and using (3.16), we obtain

$$r^{1-2k}\cdot\nabla\delta=0. \tag{3.25}$$

Formula (3.25) coincides with formula (1.8).

3.3. The multiplicative distributional product of $(r-c)^{-k}\cdot\nabla(\triangle^j\delta)$. To give a sense to the multiplicative distributional product of $(r-c)^{-k}\cdot\nabla(\triangle^j\delta)$, we must study the cases $(r-c)^{-2k}\cdot\nabla(\triangle^j\delta)$ and $(r-c)^{1-2k}\cdot\nabla(\triangle^j\delta)$ where ∇ is the operator defined by (1.9) and $\triangle^j$ is the iterated Laplacian operator defined by (1.10).

Theorem 3.3. *Let k be a positive integer and let j be a nonnegative integer, then the formula*

$$(r-c)^{-2k}\cdot\nabla(\triangle^j\delta)$$
$$=\sum_{l\geq 0}\binom{2k+2l-1}{2l}c^{2l}\cdot\left[\frac{-(n+2j)(2j)!}{(k+j+l+1)!2^{k+j+l+1}n(n+2)\cdots(n+2(k+j+l))}\right]$$
$$\times\left(\sum_{i=1}^{n}x_i\right)\triangle^{k+j+l+1}\delta \tag{3.26}$$

is valid if $k\neq n/2, n/2+1,\dots,n/2+s$, $s=0,1,\dots$, and

$$(r-c)^{1-2k}\cdot\nabla(\triangle^j\delta)$$
$$=\sum_{t\geq 1}\binom{2k+2t-1}{2t-1}c^{2t-1}\cdot\left[\frac{-(n+2j)(2j)!}{(k+j+t)!2^{k+j+t}n(n+2)\cdots(n+2(k+j+t))}\right]$$
$$\times\left(\sum_{i=1}^{n}x_i\right)\triangle^{k+j+t}\delta \tag{3.27}$$

if $k\neq n/2, n/2+1/2,\dots,n/2+s+1/2$, $s=0,1,\dots$ where $(r-c)^{-k}$ is defined by formula (3.29).

Proof. Using the formula

$$(1+z)^{\lambda}=\sum_{l\geq 0}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-l)}\frac{z^l}{l!} \tag{3.28}$$

if $|z|<1$ [4, Volume I, page 101, formulae (1), (2), and (4)], we have

$$(r-c)^{\lambda}=\sum_{l\geq 0}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-l)}\frac{(-c)^l}{l!}r^{\lambda-l} \tag{3.29}$$

if $c<r$.

Letting $\lambda=-h$, $h=1,2,\dots$, and $h\neq -n,-n-2,\dots,-n-2s$, $s=0,1,2,\dots$ in (3.29) and using the formula

$$\frac{\Gamma(\lambda+1)}{l!\,\Gamma(\lambda+1-l)}=\frac{(-1)^l\Gamma(-\lambda+l)}{l!\,\Gamma(-\lambda)}=(-1)^l\binom{-\lambda+l-1}{l}, \tag{3.30}$$

we have

$$(r-c)^{-h}=\sum_{l\geq 0}\binom{h+l-1}{l}\frac{c^l}{l!}r^{-h-l} \tag{3.31}$$

if $c<r$ and $h\neq -n,-n-2,\dots,-n-2s$, $s=0,1,2,\dots$.

Now, letting $h = 2k$ in (3.31), we have

$$
\begin{aligned}
(r-c)^{-2k}\cdot\nabla(\triangle^j\delta) &= \sum_{l\ge 0}\binom{2k+l-1}{l}c^l\left(r^{-2k-l}\cdot\nabla(\triangle^j\delta)\right)\\
&= \sum_{l\ge 0}\binom{2k+2l-1}{2l}c^{2l}\left(r^{-2(k+l)}\cdot\nabla(\triangle^j\delta)\right)\\
&\quad+\sum_{t\ge 1}\binom{2k+2t-2}{2t-1}c^{2t-1}\left(r^{1-2(t+k)}\cdot\nabla(\triangle^j\delta)\right).
\end{aligned}\tag{3.32}
$$

From (3.32), using (3.7), (3.19), and (3.25), we obtain the formula

$$
\begin{aligned}
&(r-c)^{-2k}\cdot\nabla(\triangle^j\delta)\\
&= \sum_{l\ge 0}\binom{2k+2l-1}{2l}c^{2l}\left(r^{-2(k+l)}\cdot\nabla(\triangle^j\delta)\right)\\
&= \sum_{l\ge 0}\binom{2k+2l-1}{2l}c^{2l}\cdot\left[\frac{-(n+2j)(2j)!}{(k+j+l+1)!2^{k+j+l+1}n(n+2)\cdots(n+2(k+j+l))}\right]\\
&\quad\times\left(\sum_{i=1}^{n}x_i\right)\triangle^{k+j+l+1}\delta
\end{aligned}\tag{3.33}
$$

if $k \ne n/2, n/2+1,\dots,n/2+s$, $s = 0,1,\dots$. Formula (3.33) coincides with (3.26).

Similarly, letting $h = 2k-1$ in (3.31) and using (3.8) and (3.18), we have

$$
\begin{aligned}
&(r-c)^{1-2k}\cdot\nabla(\triangle^j\delta)\\
&\quad= \sum_{l\ge 0}\binom{2k+l-2}{l}c^l\left(r^{1-2k-l}\cdot\nabla(\triangle^j\delta)\right)\\
&\quad= \sum_{l\ge 0}\binom{2k+2l}{2l}c^{2l}\left(r^{1-2(k+l)}\cdot\nabla(\triangle^j\delta)\right)\\
&\qquad+\sum_{t\ge 1}\binom{2k+2t-3}{2t-1}c^{2t-1}\left(r^{-2(k+t-1)}\cdot\nabla(\triangle^j\delta)\right)\\
&\quad= \sum_{t\ge 1}\binom{2k+2t-3}{2t-1}c^{2t-1}\left(r^{-2(k+t-1)}\cdot\nabla(\triangle^j\delta)\right)\\
&\qquad\cdot\left[\frac{-(n+2j)(2j)!}{(k+j+t)!2^{k+j+t}n(n+2)\cdots(n+2(k+j+t))}\right]\left(\sum_{i=1}^{n}x_i\right)\triangle^{k+j+t}\delta
\end{aligned}\tag{3.34}
$$

if $k \ne n/2, n/2+1/2,\dots,n/2+s+1/2$, $s = 0,1,\dots$. Formula (3.31) coincides with (3.27). □

It is clear that letting $c = 0$ in (3.26) and (3.27), we obtain formulae (3.7) and (3.18), respectively.

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References

[1] M. A. Aguirre Téllez, *Some multiplicative and convolution products between* $\delta^{(n+2j-1)}(r)$ *and* $\Delta^j\delta(x)$, Math. Balkanica (N.S.) **12** (1998), no. 1-2, 137–149.

[2] ———, *Distributional convolution product between the* k*-th derivative of Dirac's delta in* $|x|^2 - m^2$, Integral Transform. Spec. Funct. **10** (2000), no. 1, 71–80.

[3] M. A. Aguirre Téllez and C. Marinelli, *The series expansion of* $\delta^{(k)}(r-c)$, Math. Notae **35** (1991), 53–61.

[4] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions. Vols. I, II*, McGraw-Hill, New York, 1953.

[5] I. M. Gel'fand and G. E. Shilov, *Generalized Functions.*, Academic Press, New York, 1964.

[6] C. K. Li, *The product of* r^{-k} *and* $\nabla\delta$ *on* $\mathbb{R}^m$, Int. J. Math. Math. Sci. **24** (2000), no. 6, 361–369.

[7] L. Schwartz, *Théorie des distributions*, Hermann, Paris, 1996 (French).

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