

A MORE GENERALIZED GRONWALL-LIKE INTEGRAL INEQUALITY WITH APPLICATIONS

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This paper deals with a new Gronwall-like integral inequality which is a generalization of integral inequalities proved by Engler (1989) and Pachpatte (1992). The new Gronwall-like integral inequality can be used in various problems in the theory of certain class of ordinary and integral equations.

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1. Introduction. It is well known that integral inequalities play a very crucial role in the study of differential equations, integral equations, functional-differential equations, and integro-differential equations. Besides the famous Gronwall-Bellman inequality and its first nonlinear generalization by Bihari (see Bellman and Cooke [1]), there are several other very useful Gronwall-like inequalities. Haraux [3, Corollary 16, page 139] derived one Gronwall-like inequality and used it to prove the existence of solutions of wave equations with logarithmic nonlinearities. On the other hand, Engler [2] utilized the following slight variant of inequality due to Haraux [3, page 139] in the study of global regular solutions for the dynamic antiplane shear problem in nonlinear viscoelasticity.

LEMMA 1.1. *Let $c > 0$ and $a \in L^1(I = [0, T], \mathbb{R}_+ = [0, \infty))$, and assume that the function $w : I \rightarrow [1, \infty)$ satisfies*

$$w(t) \leq c \left(1 + \int_0^t a(s) w(s) \log w(s) ds \right), \quad 0 \leq t \leq T, \quad (1.1)$$

then

$$w(t) \leq c \exp \left(\int_0^t a(s) ds \right), \quad 0 \leq t \leq T. \quad (1.2)$$

Pachpatte [4] obtained one generalization of Lemma 1.1 which can be stated as follows.

LEMMA 1.2. Let $p_i \in L^1(I, \mathbb{R}_0 = (0, \infty))$ and $p \in L^1(I, \mathbb{R}_+)$. Let g be a continuously differentiable function defined on \mathbb{R}_+ , and $g > 0$ and $g' \geq 0$ on \mathbb{R}_0 . If $u : I \rightarrow \mathbb{R}_1 = [1, \infty)$ satisfies

$$u(t) \leq u_0 + H[t, p_1, p_2, \dots, p_{n-1}, pu g(\log u)], \quad t \in [0, T], \quad (1.3)$$

where $u_0 \geq 1$ is a constant,

$$\begin{aligned} & H[t, p_1, p_2, \dots, p_{n-1}, pu g(\log u)] \\ &= \int_0^t p_1(t_1) \int_0^{t_1} p_2(t_2) \cdots \int_0^{t_{n-2}} p_{n-1}(t_{n-1}) \int_0^{t_{n-1}} pu g(\log u) dt_n \cdots dt_2 dt_1, \end{aligned} \quad (1.4)$$

then, for $0 \leq t \leq t_1$, $t, t_1 \in [0, T]$,

$$u(t) \leq \exp G^{-1} \left[G(\log u_0) + H[t, p_1, p_2, \dots, p_{n-1}, p] \right], \quad (1.5)$$

where

$$G(r) = \int_{r_0}^r \frac{ds}{g(s)}, \quad r_0 > 0, \quad r \geq r_0, \quad (1.6)$$

G^{-1} is the inverse of G and t_1 is chosen so that

$$G(\log u_0) + H[t, p_1, p_2, \dots, p_{n-1}, p] \in \text{Dom}(G^{-1}) \quad (1.7)$$

for $t \in [0, t_1]$, $t_1 \in [0, T]$.

The aim of the present paper is to establish a new generalization of all the inequalities discussed in the above lemmas. One application example is also included.

2. Main results. For convenience, we give some basic notations and definitions which will be used in our subsequent discussion. Let $I = [0, T]$, $T > 0$, be finite but can be arbitrarily large. Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_0 = (0, \infty)$, and $\mathbb{R}_1 = [1, \infty)$. We denote by $L^1(I, \mathbb{R})$ the class of all measurable functions $p(t)$ defined on the set I and with range in the set \mathbb{R} with satisfying $\int_0^T |p(t)| dt < \infty$, and denote by $C^k(M, S)$ the class of all k -times continuously differentiable functions on the set M with range in the set S . We define the differential operators L_i , $0 \leq i \leq n$, by

$$L_0 x(t) = x(t), \quad L_i x(t) = \frac{1}{p_i(t)} \frac{d}{dt} (L_{i-1} x(t)), \quad 1 \leq i \leq n, \quad (2.1)$$

with $p_n(t) = 1$, where $x(t)$ and $p_i(t) > 0$ are some functions defined on I . For $t \in I$ and some functions $q_j(t) > 0, j = 1, \dots, n - 1$ and $q(t) \geq 0$ defined on I , we define

$$H[t, q_1, q_2, \dots, q_{n-1}, q] = \int_0^t q_1(t_1) \int_0^{t_1} q_2(t_2) \cdots \int_0^{t_{n-1}} q(t_n) dt_n dt_{n-1} \cdots dt_1, \tag{2.2}$$

where $t_0 = t$.

A more generalized version of the inequality appearing in [Lemma 1.2](#) is given in the following theorem.

THEOREM 2.1. *Let $p_i \in L^1(I, \mathbb{R}_0)$ and $p \in L^1(I, \mathbb{R}_+)$. Let further $g(t) \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ with satisfying $g > 0$ and $g' \geq 0$ on \mathbb{R}_0 and $\varphi \in C^2(\mathbb{R}_+, \mathbb{R}_+)$ with $\varphi'(u) > 0$ for $u > 0$ and nondecreasing. If $u : I \rightarrow \mathbb{R}_+$ satisfies*

$$\varphi(u(t)) \leq u_0 + H[t, p_1, p_2, \dots, p_{n-1}, p\varphi'(u)g(u)], \quad t \in I, \tag{2.3}$$

where $u_0 \geq 0$ is a constant, then, for $t \in I_1 = [0, t_1] \subset I$,

$$u(t) \leq G^{-1}[G(\varphi^{-1}(u_0)) + H[t, p_1, p_2, \dots, p_{n-1}, p]], \quad t \in I_1, \tag{2.4}$$

where

$$G(r) = \int_{r_0}^r \frac{ds}{g(s)}, \quad r_0 > 0, r \geq r_0, \tag{2.5}$$

and G^{-1} is the inverse function of G and t_1 is chosen so that

$$G(\varphi^{-1}(u_0)) + H[t, p_1, p_2, \dots, p_{n-1}, p] \in \text{Dom}(G^{-1}) \tag{2.6}$$

for $t \in I_1$.

PROOF. Let $\varepsilon > 0$ be an arbitrary small constant and define on I a nondecreasing function

$$v_\varepsilon(t) = u_0 + \varepsilon + H[t, p_1, p_2, \dots, p_{n-1}, p\varphi'(u)g(u)]. \tag{2.7}$$

From (2.3) and (2.7), we have

$$u(t) \leq \varphi^{-1}(v_\varepsilon(t)), \quad t \in I, \tag{2.8}$$

$$v_\varepsilon(0) = u_0 + \varepsilon > 0,$$

$$L_n v_\varepsilon(t) = p(t)\varphi'(u(t))g(u(t)). \tag{2.9}$$

Since φ' and g are nondecreasing, by (2.8) and (2.9), we observe that

$$L_n v_\varepsilon(t) \leq p(t)\varphi'[\varphi^{-1}(v_\varepsilon(t))]g[\varphi^{-1}(v_\varepsilon(t))]. \tag{2.10}$$

Using the fact that $v_\varepsilon(t)$ is positive,

$$\frac{d}{dt}(\varphi'[\varphi^{-1}(v_\varepsilon(t))]) = \varphi''[\varphi^{-1}(v_\varepsilon(t))] \times \frac{1}{\varphi'(v_\varepsilon(t))} \times \frac{dv_\varepsilon(t)}{dt} \geq 0, \tag{2.11}$$

and $L_n v_\varepsilon(t) \geq 0$ for $t \in I$, it follows from (2.10) that

$$\begin{aligned} \frac{L_n v_\varepsilon(t)}{\varphi'[\varphi^{-1}(v_\varepsilon(t))]} &\leq p(t)g[\varphi^{-1}(v_\varepsilon(t))] \\ &+ \frac{L_{n-1} v_\varepsilon(t) \times (d/dt)(\varphi'[\varphi^{-1}(v_\varepsilon(t))])}{\varphi'^2[\varphi^{-1}(v_\varepsilon(t))]}, \end{aligned} \tag{2.12}$$

that is,

$$\frac{d}{dt} \left(\frac{L_{n-1} v_\varepsilon(t)}{\varphi'[\varphi^{-1}(v_\varepsilon(t))]} \right) \leq p(t)g[\varphi^{-1}(v_\varepsilon(t))]. \tag{2.13}$$

Integrating (2.13) from 0 to t and using the fact that $L_{n-1} v_\varepsilon(0) = 0$ give

$$\frac{L_{n-1} v_\varepsilon(t)}{\varphi'[\varphi^{-1}(v_\varepsilon(t))]} \leq \int_0^t p(t_n)g[\varphi^{-1}(v_\varepsilon(t_n))] dt_n. \tag{2.14}$$

It also follows from (2.14) that

$$\frac{d}{dt} \left(\frac{L_{n-2} v_\varepsilon(t)}{\varphi'[\varphi^{-1}(v_\varepsilon(t))]} \right) \leq p_{n-1}(t) \int_0^t p(t_n)g[\varphi^{-1}(v_\varepsilon(t_n))] dt_n, \tag{2.15}$$

which, upon integrating from 0 to t and using the fact that $L_{n-2} v_\varepsilon(0) = 0$, leads to

$$\frac{L_{n-2} v_\varepsilon(t)}{\varphi'[\varphi^{-1}(v_\varepsilon(t))]} \leq \int_0^t p_{n-1}(t_{n-1}) \int_0^{t_{n-1}} p(t_n)g[\varphi^{-1}(v_\varepsilon(t_n))] dt_n dt_{n-1}. \tag{2.16}$$

Repeating the above argument successively, we obtain

$$\begin{aligned} \frac{(d/dt)v_\varepsilon(t)}{\varphi'[\varphi^{-1}(v_\varepsilon(t))]} &\leq p_1(t) \int_0^t p_2(t_2) \int_0^{t_2} p_3(t_3) \cdots \int_0^{t_{n-2}} p_{n-1}(t_{n-1}) \\ &\cdot \int_0^{t_{n-1}} p(t_n)g[\varphi^{-1}(v_\varepsilon(t_n))] dt_n dt_{n-1} \cdots dt_3 dt_2. \end{aligned} \tag{2.17}$$

For any invertible and continuously differentiable function $a(t)$, by changing the variable $\eta = a^{-1}(\xi)$, we have

$$\int \frac{d\xi}{a'[a^{-1}(\xi)]} = \int \frac{a'(\eta)}{a'(\eta)} d\eta = \eta + c = a^{-1}(\xi) + c. \tag{2.18}$$

Using the above fact and integrating (2.17) from 0 to t , we obtain

$$\varphi^{-1}(v_\varepsilon(t)) \leq \varphi^{-1}(u_0 + \varepsilon) + H[t, p_1, p_2, \dots, p_{n-1}, p g[\varphi^{-1}(v_\varepsilon(t))]]. \tag{2.19}$$

Define a function $w(t)$ by

$$w(t) = \varphi^{-1}(u_0 + \varepsilon) + H[t, p_1, p_2, \dots, p_{n-1}, p g[\varphi^{-1}(v_\varepsilon(t))]], \tag{2.20}$$

then

$$\varphi^{-1}(v_\varepsilon(t)) \leq w(t), \quad t \in I, \tag{2.21}$$

and $w(0) = \varphi^{-1}(u_0 + \varepsilon) > 0$.

It follows from (2.19) and (2.21) that

$$L_n w(t) = p(t)g[\varphi^{-1}(v_\varepsilon(t))] \leq p(t)g(w(t)). \tag{2.22}$$

From (2.22) and using the fact that $g(w(t))$ is positive, $g'(w(t)) \geq 0$, $L_{n-1} w(t) \geq 0$ for $t \in I$, we observe that

$$\frac{L_n w(t)}{g(w(t))} \leq p(t) + \frac{[(d/dt)g(w(t))] \cdot L_{n-1} w(t)}{g^2(w(t))}, \tag{2.23}$$

that is,

$$\frac{d}{dt} \left[\frac{L_{n-1} w(t)}{g(w(t))} \right] \leq p(t). \tag{2.24}$$

Starting with (2.24) and using the same steps as used from (2.13) to (2.17), we derive that

$$\begin{aligned} \frac{(d/dt)w(t)}{g(w(t))} &\leq p_1(t) \int_0^t p_2(t_2) \int_0^{t_2} p_3(t_3) \cdots \int_0^{t_{n-2}} p_{n-1}(t_{n-1}) \\ &\quad \cdot \int_0^{t_{n-1}} p(t_n) dt_n dt_{n-1} \cdots dt_3 dt_2. \end{aligned} \tag{2.25}$$

Now, by the definition of G , keeping $t = t_1$ and integrating both sides of (2.25) from 0 to t , give

$$G(w(t)) \leq G(w(0)) + H[t, p_1, p_2, \dots, p_{n-1}, p], \tag{2.26}$$

that is,

$$w(t) \leq G^{-1}[G(\varphi^{-1}(u_0 + \varepsilon)) + H[t, p_1, p_2, \dots, p_{n-1}, p]], \quad t \in I_1. \tag{2.27}$$

The desired bound in (2.4) now follows from (2.8), (2.21), and the last inequality (2.27), together with the limit $\varepsilon \rightarrow 0$. □

COROLLARY 2.2. *Suppose that the functions p, p_1, \dots, p_{n-1} and g are defined as in Theorem 2.1, and $u_0 \geq 1$ and $k > 0$ are constants. If $u : I \rightarrow \mathbb{R}_1$ satisfies*

$$u^k(t) \leq u_0 + H[t, p_1, p_2, \dots, p_{n-1}, pu^k g(\log u)], \quad t \in I, \tag{2.28}$$

then, for $t \in I_2$,

$$u(t) \leq \exp G^{-1}\left[G\left(\frac{1}{k} \log u_0\right) + H\left[t, p_1, p_2, \dots, p_{n-1}, \frac{1}{k} p\right]\right], \tag{2.29}$$

where G and G^{-1} are defined as in Theorem 2.1 and $I_2 = [0, t_2] \subset I$, t_2 is chosen so that

$$G\left(\frac{1}{k} \log u_0\right) + H\left[t, p_1, p_2, \dots, p_{n-1}, \frac{1}{k} p\right] \in \text{Dom}(G^{-1}). \tag{2.30}$$

PROOF. Changing the variable $u = \exp(v)$ in (2.28) leads to

$$\begin{aligned} \exp(kv) &\leq u_0 + H[t, p_1, p_2, \dots, p_{n-1}, p \exp(kv) g(v)] \\ &= u_0 + H\left[t, p_1, p_2, \dots, p_{n-1}, \frac{p}{k} (\exp(kv))' g(v)\right]. \end{aligned} \tag{2.31}$$

This inequality is a special case of Theorem 2.1 when $\varphi = \exp(kv)$. By Theorem 2.1, we derive from inequality (2.31) that

$$v \leq G^{-1}\left[G\left(\frac{1}{k} \log u_0\right) + H\left[t, p_1, p_2, \dots, p_{n-1}, \frac{1}{k} p\right]\right], \quad t \in I_2. \tag{2.32}$$

This follows inequality (2.29). □

REMARK 2.3. When $k = 1$ in Corollary 2.2, we derive the assertion of Lemma 1.2.

Letting $\varphi = u^k$ ($k > 1$ is a constant) in [Theorem 2.1](#) leads to the following corollary.

COROLLARY 2.4. *Suppose that the functions $u, p, p_1, \dots, p_{n-1}$ and g are defined as in [Theorem 2.1](#) and $k > 1$ is a constant, then the inequality*

$$u^k(t) \leq u_0 + H[t, p_1, p_2, \dots, p_{n-1}, kp u^{k-1}g(u)], \quad t \in I, \tag{2.33}$$

implies

$$u(t) \leq G^{-1}[G(u_0^{1/k}) + H[t, p_1, p_2, \dots, p_{n-1}, kp]], \quad t \in I_3, \tag{2.34}$$

where G and G^{-1} are defined as in [Theorem 2.1](#) and $I_3 = [0, t_3] \subset I$, t_3 is chosen so that

$$G(u_0^{1/k}) + H[t, p_1, p_2, \dots, p_{n-1}, kp] \in \text{Dom}(G^{-1}). \tag{2.35}$$

REMARK 2.5. Setting by $n = 1, k = 2, g \equiv 1$ in [Corollary 2.4](#), we arrive at Ou-Iang’s integral inequality given in [\[5\]](#).

REMARK 2.6. By choosing other suitable special functions to φ , we get other interesting inequalities which could not be derived from [Lemma 1.2](#).

3. Application. Consider the differential equation

$$\begin{aligned} L_n x^k(t) &= p(t)x^{k-1}(t)f(x(t)), \quad t \in I, \\ L_i x^k(0) &= C_{i-1}, \quad i = 1, 2, \dots, n, \end{aligned} \tag{3.1}$$

where $k > 1$ and $C_{i-1}, 1 \leq i \leq n$ are constants, p and $f \in C(I, \mathbb{R})$, and L_n is defined as in [Section 2](#). It is easy to observe that [\(3.1\)](#) is equivalent to the integral equation

$$x^k(t) = b(t) + H[t, p_1, p_2, \dots, p_{n-1}, px^{k-1}f(x)], \quad t \in I, \tag{3.2}$$

where

$$b(t) = C_0 + \sum_{i=1}^{n-1} C_i H[t, p_1, \dots, p_i]. \tag{3.3}$$

If $|b(t)| \leq u_0$ and $|f(u)| \leq g(|u|)$, where u_0 and g are defined as in [Theorem 2.1](#), then we derive from [\(3.2\)](#) that

$$|x(t)|^k \leq u_0 + H[t, p_1, p_2, \dots, p_{n-1}, |p||x|^{k-1}g(|x|)], \quad t \in I. \tag{3.4}$$

An application of [Corollary 2.4](#) to inequality [\(3.4\)](#) yields

$$|x(t)| \leq G^{-1} \left[G(u_0^{1/k}) + H \left[t, p_1, p_2, \dots, p_{n-1}, \frac{1}{k} |p| \right] \right], \quad t \in I_3 \cap J(x), \quad (3.5)$$

where $J(x)$ denotes the maximal existent interval of $x(t)$.

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REFERENCES

- [1] R. Bellman and K. L. Cooke, *Differential-Difference Equations*, Academic Press, New York, 1963.
- [2] H. Engler, *Global regular solutions for the dynamic antiplane shear problem in nonlinear viscoelasticity*, *Math. Z.* **202** (1989), no. 2, 251–259.
- [3] A. Haraux, *Nonlinear Evolution Equations—Global Behavior of Solutions*, Lecture Notes in Mathematics, vol. 841, Springer-Verlag, Berlin, 1981.
- [4] B. G. Pachpatte, *On Gronwall-like integral inequalities*, *Indian J. Pure Appl. Math.* **23** (1992), no. 2, 131–140.
- [5] Ou Yang-Liang, *The boundedness of solutions of linear differential equations $y'' + A(t)y = 0$* , *Advances in Math.* **3** (1957), 409–415 (Chinese).

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