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SOME SUBMERSIONS OF CR-HYPERSURFACES OF KAEHLER-EINSTEIN MANIFOLD

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The Riemannian submersions of a CR-hypersurface M of a Kaehler-Einstein manifold $\tilde{M}$ are studied. If M is an extrinsic CR-hypersurface of $\tilde{M}$, then it is shown that the base space of the submersion is also a Kaehler-Einstein manifold.

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1. Introduction. The study of the Riemannian submersions $\pi : M \to B$ was initiated by O'Neill [14] and Gray [9]. This theory was very much developed in the last thirty five years. Besse's book [3, Chapter 9] is a reference work. Bejancu introduced a remarkable class of submanifolds of a Kaehler manifold that are known as CR-submanifolds (see [1, 2]). On a CR-submanifold, there are two complementary distributions D and $D^\perp$, such that D is J-invariant and $D^\perp$ is J-anti-invariant with respect to the complex structure J of the Kaehler manifold. The integrability of the anti-invariant distribution D was proved by Blair and Chen [4].

Recently, Kobayashi [10] considered the similarity between the total space of a Riemannian submersion and a CR-submanifold of a Kaehler manifold in terms of the distribution. He studied the case of a generic CR-submanifolds in a Kaehler manifold and proved that the base space is a Kaehler manifold.

In Section 3, we extend the result of Kobayashi to the general case of a CR-submanifold.

In Section 4, we study a Riemannian submersion from an extrinsic hypersurface M of a Kaehler-Einstein manifold $\tilde{M}$ onto an almost-Hermitian manifold B. In this case, we prove that the basic manifold is a Kaehler-Einstein manifold. If $\tilde{M}$ is C^{n+1}, a standard example is the Hopf fibration $S^{2n+1} \to CP^n$ equipped with the canonical metrics.

For the basic formulas of Riemannian geometry, we use [11, 12].

2. Preliminaries. Let $\tilde{M}$ be a complex m-dimensional Kaehler manifold with complex structure J and Hermitian metric $\langle \cdot, \cdot \rangle$. Bejancu [2] introduced the concept of a CR-submanifold of $\tilde{M}$ as follows: a real Riemannian manifold M, isometrically immersed in a Kaehler manifold $\tilde{M}$, is called a CR-submanifold of $\tilde{M}$ if there exists on M a differentiable holomorphic distribution D and its orthogonal complement $D^\perp$ on M is a totally real distribution, that is, $JD_x^\perp \subseteq T_x^\perp M$, where $T_x^\perp M$ is the normal space to M at $x \in M$ for any $x \in M$. It is easily seen that each real orientable hypersurface of M is a CR-submanifold. The Riemannian metric induced on M will be denoted by the same symbol $\langle \cdot, \cdot \rangle$.

Let $\tilde{\nabla}$ (resp., ∇) be the operator of covariant differentiation with respect to the Levi-Civita connection on $\tilde{M}$ (resp., M). The second-fundamental form B is given by

$$B(E,F) = \tilde{\nabla}_E F - \nabla_E F \tag{2.1}$$

for all $E, F \in \Gamma(TM)$, where $\Gamma(TM)$ is the space of differentiable vector field on M. We denote everywhere by $\Gamma(\tau)$ the space of differentiable sections of a vector bundle τ.

For a normal vector field N, that is, $N \in \Gamma(T^\perp M)$, we write

$$\tilde{\nabla}_E N = -L_N E + \nabla_E^\perp N, \tag{2.2}$$

where $-L_N E$ (resp., $\nabla_E^\perp N$) denotes the tangential (resp., normal) component of $\tilde{\nabla}_E N$.

Let μ be the orthogonal complementary vector bundle of $J(D^\perp)$ in $T^\perp M$, that is, $T^\perp M = J(D^\perp) \oplus \mu$.

It is clear that μ is a holomorphic subbundle of $T^\perp M$, that is, $J\mu = \mu$.

Definition 2.1 (Kobayashi [10])**.** Let M be a CR-submanifold of a Kaehler manifold $\tilde{M}$. A submersion from a CR-manifold M onto an almost-Hermitian manifold is a Riemannian submersion $\pi : M \to M'$ with the following conditions:

(i) $D^\perp$ is the kernel of π_*,

(ii) $\pi_* : D_x \to T_{\pi(x)} M'$ is a complex isometry for every $x \in M$.

This definition is given by Kobayashi for the case where μ is a null subbundle of $T^\perp M$ (see [10]). If $JD_x^\perp = T_x^\perp M$ for any $x \in M$, we say that M is a *generic CR-submanifold* of $\tilde{M}$ (Yano and Kon [15]). For example, any real orientable hypersurface of $\tilde{M}$ is a generic CR-submanifold of $\tilde{M}$.

Concerning the basic notions on the Riemannian submersions, see O'Neill [14] and Gray [9].

The vertical distribution of a Riemannian submersion is an integrable distribution. In our case, the distribution vertical is $D^\perp$, which is integrable according to a theorem by Blair and Chen [4].

The sections of $D^\perp$ (resp., D) are called the *vertical vector fields* (resp., the *horizontal vector fields*) of the Riemannian submersion $\pi : M \to M'$. The letters U, V, W, and W' will always denote vertical vector fields, and the letters X, Y, Z, and Z' denote horizontal vector fields. For any $E \in \mathscr{X}(M)$, υE and hE denote the vertical and horizontal components of E, respectively. A horizontal vector field X on M is said to be basic if X is π-related to a vector field X' on M'.

It is easy to see that every vector field X' on M' has a unique horizontal lift X to M, and X is basic.

Conversely, let X be a horizontal vector field and suppose that $\langle X,Y\rangle_x = \langle X,Y\rangle_y$ for all Y basic vector fields on M, for all $x,y \in \pi^{-1}(x')$, and for all $x' \in M'$. Then, the vector field X is basic. We have the following O'Neill's lemma (see [8, 14]).

Lemma 2.2. *Let X and Y be basic vector fields on M. Then, they are satisfying the following:*

(i) *the horizontal component $h[X,Y]$ of $[X,Y]$ is a basic vector field and $\pi_* h[X,Y] = [X',Y'] \circ \pi$,*

(ii) *$h(\nabla_X Y)$ is a basic vector field corresponding to $\nabla'_{X'} Y'$, where ∇' is the Levi-Civita connection on $(M', \langle , \rangle')$,*

(iii) *$[X,U] \in \Gamma(D^\perp)$ for any vertical field $U \in \Gamma(D^\perp)$.*

We recall that a Riemannian submersion $\pi : (M,g) \to (M',g')$ determines the fundamental tensor field T and A by the formulas

$$\begin{aligned} T_E F &= h\nabla_{vE} vF + v\nabla_{vE} hF, \\ A_E F &= v\nabla_{hE} hF + h\nabla_{hE} vF, \end{aligned} \tag{2.3}$$

for all $E,F \in \Gamma(TM)$ (cf. O'Neill [14] and Besse [3]).

It is easy to prove that T and A satisfy

$$T_U V = T_v U, \tag{2.4}$$

$$A_X Y = \frac{1}{2} v[X,Y], \tag{2.5}$$

for any $U,V \in \Gamma(D^\perp)$ and $X,Y \in \Gamma(D)$.

Formula (2.4) means that the restriction of T to the integrable distribution $D^\perp$ is the second-fundamental form of the fiber submanifolds in M, and (2.5) measures the integrability of the distribution D.

We have the following properties:

$$\begin{aligned} \nabla_U X &= T_U X + h\nabla_U X, \\ \nabla_X U &= v\nabla_X U + A_X U, \\ \nabla_X Y &= h\nabla_X Y + A_X Y, \end{aligned} \tag{2.6}$$

for any $X,Y \in \Gamma(\mathcal{H})$ and $U \in \Gamma(\mathcal{V})$.

3. Kaehler structure on the basic space M'. From (2.1), we have

$$\tilde{\nabla}_X Y = h\nabla_X Y + v\nabla_X Y + \overline{h}B(X,Y) + \overline{v}B(X,Y) \tag{3.1}$$

for any $X,Y \in \Gamma(D)$.

Here, we denote by h and v (resp., $\overline{h}$ and $\overline{v}$) the canonical projections on D and $D^\perp$ (resp., μ and $JD^\perp$). Define a tensor field C on M as the vertical component $v(\nabla_X Y)$ of $\nabla_X Y$ (cf. Kobayashi [10]). The tensor field C is known to be a skew-symmetric tensor field defined by Kobayashi such that

$$C(X,Y) = \frac{1}{2}v[X,Y] \tag{3.2}$$

for all $X, Y \in \Gamma(D)$.

Note that the tensor field C is the restriction of A to $\Gamma(\mathcal{H}) \times \Gamma(\mathcal{H})$.

From Definition 2.1 and Lemma 2.2, we obtain that $Jh\nabla_X Y$ (resp., $h\nabla_X JY$) is a basic vector field and corresponds to $J'\nabla'_{X'}Y'$ (resp., $\nabla'_{X'}J'Y'$) for any basic vector fields X and Y on M.

On the Kaehler manifold $\tilde{M}$, we have

$$\tilde{\nabla}_E JF = J\tilde{\nabla}_E F. \tag{3.3}$$

From (3.1) and (3.3), we obtain the following proposition.

Proposition 3.1. For any basic vector fields X and Y on M,

$$Jh\nabla_X Y = h\nabla_X JY, \tag{3.4}$$

$$JC(X,Y) = \overline{v}B(X,JY), \tag{3.5}$$

$$C(X,JY) = J\overline{v}B(X,Y), \tag{3.6}$$

$$J\overline{h}B(X,Y) = \overline{h}B(X,JY). \tag{3.7}$$

Theorem 3.2. *Let M be a CR-submanifold of a Kaehler manifold $\tilde{M}$ and $\pi : M \to M'$ be a CR-submersion of M on an almost-Hermitian manifold M'. Then, M' is a Kaehler manifold.*

Proof. From Lemma 2.2 and (3.4), we obtain that $\nabla'_{X'}J'Y' = J'\nabla'_{X'}Y'$, so that M' is a Kaehler manifold. □

Remark 3.3. Proposition 3.1 is proved for generic CR-submanifolds of $\tilde{M}$ (i.e., $\mu = 0$) in [10].

4. Riemannian submersions from extrinsic hyperspheres of Einstein-Kaehler manifolds. We recall that a totally umbilical submanifold M of a Riemannian manifold $\tilde{M}$ is a submanifold whose first-fundamental form and second-fundamental form are proportional.

The extrinsic hyperspheres are defined to be totally umbilical hypersurfaces, having nonzero parallel mean-curvature vector field (cf. Nomizu and Yano [13]). Many of the basic results concerning extrinsic spheres in Riemannian and Kaehlerian geometry were obtained by Chen [5, 6, 7].

Let M be an orientable hypersurface in a Kaehler manifold $\tilde{M}$. Then, M is an extrinsic hypersphere of $\tilde{M}$ if it satisfies

$$B(E,F) = \langle E,F\rangle H \tag{4.1}$$

for any vector fields E and F on M. Here, H denote the mean-curvature vector field of M. If we put $k = \| H \|$ (where the norm $\| \cdot \|$ is, with respect to a scalar product, induced on every tangent space to M), then k is a nonzero constant function on the extrinsic hypersphere M.

We denote by N the global unit normal vector field to M. Then, $\xi = -JN$ is a global unit vector on M such that $N = J\xi$. Let D be the maximal J-invariant subspace (with respect to J) of the tangent space T_pM for every $p \in M$. We see that M is a CR-hypersurface of M such that $TM = D \oplus D^\perp$, where $D^\perp$ is the one-dimensional anti-invariant distribution generated by the vector field ξ on M.

The anti-invariant distribution $D^\perp$ is integrable, and its leaves are totally geodesic in M (but not in $\tilde{M}$).

This is an easy consequence from Gauss and Weingarten's formulas of the leaves of $D^\perp$ in M. This means that O'Neill's tensor T vanishes on the fibres of the Riemannian submersion $\pi : M \to B$.

The main result of this section is the following theorem.

Theorem 4.1. *Let M be an orientable extrinsic hypersphere of an Kaehler-Einstein manifold $\tilde{M}$. If $\pi : M \to B$ is a CR-submersion of M on an almost-Hermitian manifold B, then B is an Kaehler-Einstein manifold.*

To prove Theorem 4.1, we need several lemmas.

Lemma 4.2. *Following the assumptions of Theorem 4.1, then*

$$\langle A_x\xi, A_y\xi\rangle = k^2\langle X,Y\rangle \tag{4.2}$$

for any horizontal vector X on M.

Proof. From Gauss's formula (2.1) and the umbilicality of M, we get $\tilde{\nabla}_X\xi = \nabla_X\xi$ for any vector field X on M. Then, we have

$$\langle \tilde{\nabla}_X JN, Y\rangle = \langle \nabla_X\xi, Y\rangle = \langle h\nabla_X\xi, Y\rangle = \langle A_X\xi, Y\rangle. \tag{4.3}$$

On the other hand, $\tilde{M}$ is a Kaehler manifold, so that ∇ commute with J:

$$\begin{aligned}\langle \tilde{\nabla}_X JN, Y\rangle &= \langle J\tilde{\nabla}_X N, Y\rangle = -\langle \tilde{\nabla}_X N, JY\rangle = \langle B(X,JY),N\rangle \\ &= \langle G(X,JY)H,N\rangle = k\langle X,JY\rangle.\end{aligned} \tag{4.4}$$

Consequently,

$$\langle A_X\xi, A_Y\xi\rangle = k\langle X, JA_Y\xi\rangle = -k\langle JX, A_Y\xi\rangle = k^2\langle X, Y\rangle. \tag{4.5}$$

□

Lemma 4.3. *Following the assumptions of Theorem 4.1, then*

$$\langle A_XY, A_ZW\rangle = k^2\langle X, JY\rangle\langle Z, JW\rangle \tag{4.6}$$

for any horizontal vector fields on M.

Proof. We say that A_XY is a vertical vector field, hence

$$A_XY = \langle A_XY, \xi\rangle\xi. \tag{4.7}$$

Then,

$$\langle A_XY, A_ZW\rangle = \langle A_XY, \xi\rangle\langle A_ZW, \xi\rangle = k^2\langle X, JY\rangle\langle Z, JW\rangle. \tag{4.8}$$

□

Lemma 4.4. *Following the assumptions of Theorem 4.1, then*

$$\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + k^2\{\langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle\}, \tag{4.9}$$

where $\tilde{R}$ and R are the curvature tensor on $\tilde{M}$ and M, respectively.

Proof. We have the Gauss equation

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + \langle B(X,Z), B(Y,W)\rangle \\ - \langle B(Y,Z), B(X,W)\rangle.\end{aligned} \tag{4.10}$$

Using the umbilicality condition, we get (4.9). □

Lemma 4.5. *For any horizontal vector fields X and Y on M,*

$$\tilde{R}(\xi, X, Y, \xi) = 0, \qquad \tilde{R}(\xi, JX, JY, \xi) = 0. \tag{4.11}$$

Proof. For a Riemannian submersion with totally geodesic fibres, the following formula is known:

$$\tilde{R}(X,V,Y,U) = \langle(\nabla_V A)(X,Y), U\rangle + \langle A_XV, A_YU\rangle. \tag{4.12}$$

On the other hand, the first term on the right part is skew-symmetric with respect to the vertical vector fields V and U. From (4.12) and (4.9), we obtain (4.11). □

Proof of Theorem 4.1. *For the horizontal vector fields X, Y, Z, and W on M, we have the following equation of O'Neill:*

$$R(X,Y,Z,W) = R'(X',Y'\cdot Z',W') - 2\langle A_X Y, A_Z W\rangle + \langle A_Y Z, A_X W\rangle - \langle A_X Z, A_Y W\rangle \tag{4.13}$$

(see [3, 14]).

By (4.9) and (4.11), we get the following formula that connects the curvature of M' to the curvature of the Kaehler manifold $\tilde{M}$:

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = R'(X',Y',Z',W') &\\ - k^2\{\langle X,JZ\rangle\langle Y,JW\rangle - \langle X,JW\rangle\langle Y,JZ\rangle &\\ + 2\langle X,JY\rangle\langle Z,JW\rangle\} &\\ - k^2\{\langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle\}. &\end{aligned} \tag{4.14}$$

Let $(e_1,\dots,e_p;Je_1,\dots,Jl_p)$ be a local J-frame of basic vector fields for the horizontal distribution D. Then, $(e_1,\dots,e'_p;J'e_1,\dots,J'e_p)$ is a local J'-frame if $\pi_{\text{star}} e_i = e'_i$ on the Kaehler manifold B.

Using the above lemmas, from (4.14) by a straightforward calculation, we conclude that B is a Kaehler-Einstein manifold if $\tilde{M}$ is a Kaehler-Einstein manifold.

□

Corollary 4.6. *Let $\tilde{M}$ be a complex-form space and M an orientable CR-hypersurface of $\tilde{M}$. Then, the base space of submersion $\pi : M \to B$ is also a complex-form space.*

Proof. The corollary follows by straightforward calculation making use of (4.14). □

Example 4.7. Let S^{2n+1} be the standard hypersphere in C^{n+1}. Then, S^{2n+1} is an extrinsic hypersphere in C^{n+1}, and we have the Hopf fibration $\pi : S^{2n+1} \to CP^n$ equipped with the canonical metrics.

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