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ON CENTRALIZERS OF ELEMENTS OF GROUPS ACTING ON TREES WITH INVERSIONS

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A subgroup H of a group G is called malnormal in G if it satisfies the condition that if $g \in G$ and $h \in H$, $h \neq 1$ such that $ghg^{-1} \in H$, then $g \in H$. In this paper, we show that if G is a group acting on a tree X with inversions such that each edge stabilizer is malnormal in G, then the centralizer $C(g)$ of each nontrivial element g of G is in a vertex stabilizer if g is in that vertex stabilizer. If g is not in any vertex stabilizer, then $C(g)$ is an infinite cyclic if g does not transfer an edge of X to its inverse. Otherwise, $C(g)$ is a finite cyclic of order 2.

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1. Introduction. There are many groups with the property that centralizers of nontrivial elements are cyclic. For example, the centralizers of the nontrivial elements (see [3, Problem 7, page 42]) of free groups are infinite cyclic. It has been shown in [3, Problem 28, page 196] that the centralizers of the nontrivial elements of the free product of groups are in a conjugate of a factor or infinite cyclic. In [1, Theorem 1], Karrass and Solitar proved that the centralizers of nontrivial elements of the free product of two groups with a malnormal amalgamated subgroup are in a conjugate of a factor or infinite cyclic. In this paper, we generalize such results to groups acting on trees with inversions as follows. If G is a group acting on a tree X with inversions such that the stabilizer G_x of every edge x of X is a malnormal subgroup in G, then the centralizers $C(g)$ of every nontrivial element g of G is in a vertex stabilizer G_v of a vertex v of X if g is in G_v. If g is not in any vertex stabilizer of G, then $C(g)$ is an infinite cyclic subgroup of G if g does not transfer any edge of X to its inverse. Otherwise, $C(g)$ is a finite cyclic subgroup of G of order 2.

This paper is divided into five sections. In Section 2, we introduce the concepts of graphs and the actions of groups on graphs. In Section 3, we have a summary of the structure of groups acting on trees with inversions and elementary results. In Section 4, we discuss the structure of the centralizers of the elements of groups acting on trees with inversions. Section 5 is an application of the results in Section 4.

2. Basic concepts. We begin by giving preliminary definitions.

By a graph X, we understand a pair of disjoint sets $V(X)$ and $E(X)$ with $V(X)$ nonempty together with a mapping $E(X) \to V(X)\times V(X)$, $y \to (o(y),t(y))$, and a mapping $E(X) \to E(X)$, $y \to \bar{y}$, satisfying the conditions that $\bar{\bar{y}} = y$ and $o(\bar{y}) = t(y)$, for all $y \in E(X)$. The case $\bar{y} = y$ is possible for some $y \in E(X)$. For $y \in E(X)$, $o(y)$ and $t(y)$ are called the ends of y, and $\bar{y}$ is called the inverse of y. There are obvious definitions of subgraphs, trees, morphisms of graphs, and $\mathrm{Aut}(X)$, the set of all automorphisms of the graph X which is a group under the composition of morphisms of graphs. For more details, see [4, 5]. We say that a group G acts on a graph X if there is a group homomorphism $\phi : G \to \mathrm{Aut}(X)$. If $x \in X$ (vertex or edge) and $g \in G$, we write $g(x)$ for $(\phi(g))(x)$. If $y \in E(X)$ and $g \in G$, then $g(o(y)) = o(g(y))$, $g(t(y)) = t(g(y))$, and $g(\bar{y}) = \overline{g(y)}$. The case $g(y) = \bar{y}$ for some $g \in G$ and some $y \in E(X)$ may occur. That is, G acts with inversions on X.

We have the following notations related to the action of the group G on the graph X.

(1) If $x \in X$ (vertex or edge), we define $G(x) = \{g(x) : g \in G\}$, and this set is called the orbit containing x.

(2) If $x,y \in X$, we define $G(x,y) = \{g \in G : g(x) = y\}$ and $G(x,x) = G_x$, the stabilizer of x. Thus, $G(x,y) \neq \emptyset$ if and only if x and y are in the same G orbit. It is clear that if $v \in V(X)$, $y \in E(X)$, and $u \in \{o(y),t(y)\}$, then $G(v,y) = \emptyset$, $G_{\bar{y}} = G_y$, and $G_y \leq G_u$.

3. The structure of groups acting on trees with inversions.

In this section, we summarize the structure of groups acting on trees with inversions obtained by [4].

Definition 3.1. Let G be a group acting on a tree X, and T and Y two subtrees of X such that $T \subseteq Y$. Then, T is called a tree of representatives for the action of G on X if T contains exactly one vertex from each G vertex orbit, and Y is called a fundamental domain for the action G on X, if each edge of Y has at least one end in T, and Y contains exactly one edge y (say) from each G edge orbit such that $G(\bar{y},y) = \emptyset$, and exactly one pair x and $\bar{x}$ from each G-edge orbit such that $G(\bar{x},x) \neq \emptyset$. It is clear that the properties of T and Y imply that if u and v are two vertices of T such that $G(u,v) \neq \emptyset$, and if x and y are two edges of Y such that $G(x,y) \neq \emptyset$, then $u = v$ and $x = y$ or $x = \bar{y}$.

Let T and Y be as above. Define the following subsets Y_0, Y_1, and Y_2 of edges of Y as follows:

(1) $Y_0 = E(T)$, the set of edges of T,

(2) $Y_1 = \{y \in E(Y) : o(y) \in V(T),\ t(y) \notin V(T),\ G(\bar{y},y) = \emptyset\}$,

(3) $Y_2 = \{x \in E(Y) : o(x) \in V(T),\ t(x) \notin V(T),\ G(\bar{x},x) \neq \emptyset\}$.

It is clear that G acts with inversions on X if and only if $Y_2 \neq \emptyset$.

For the rest of this section, G will be a group acting on a tree X with inversions, T will be a tree of representatives for the action of the group G on X, and Y will be a fundamental domain for the action of G on X such that $T \subseteq Y$. We have the following definitions.

Definition 3.2. For each vertex v of X, define v^* to be the unique vertex of T such that $G(v,v^*) \neq \emptyset$. That is, v and v^* are in the same G vertex orbit. It is clear that if v is a vertex of T, then $v^* = v$ and, in general, for any two vertices u and v of X such that $G(u,v) \neq \emptyset$, we have $u^* = v^*$, and G_u and G_v are conjugate by an element of G. That is, for every element b of G_u, there exist g of G and a of G_v such that $b = gag^{-1}$.

Definition 3.3. For each edge y of $Y_0 \cup Y_1 \cup Y_2$, define $[y]$ to be an element of $G(t(y),(t(y))^*)$. That is, $[y]$ satisfies the condition that $[y]((t(y))^*) = t(y)$, and to be chosen as follows:

$$[y] = 1 \quad \text{if } y \in Y_0, \qquad [y](y) = \bar{y} \quad \text{if } y \in Y_2. \tag{3.1}$$

Define $[\bar{y}]$ to be the element

$$[\bar{y}] = \begin{cases} [y] & \text{if } y \in Y_0 \cup Y_2, \\ [y]^{-1} & \text{if } y \in Y_1. \end{cases} \tag{3.2}$$

From above we see that $[y](y) \neq \bar{y}$ if $y \in Y_0 \cup Y_1$ and $[y](y) = \bar{y}$ if $y \in Y_2$.

A group is termed a *quasifree* group if it is a free product of copies of C_∞ and C_2, where C_∞ denotes infinite cyclic group and C_2 a cyclic group of order 2.

The following are examples of quasifree groups:

(1) every free group is a quasifree group. That is, a free product of copies of C_∞ and a zero number of copies of C_2;

(2) the group of the presentation $\langle x,y,z \mid z^2 = 1\rangle \cong C_\infty * C_\infty * C_2$ is a quasifree group;

(3) the infinite dihedral group $\langle x,y \mid x^2 = 1,\ y^2 = 1\rangle \cong C_2 * C_2$ is a quasifree group.

Lemma 3.4. *Let G, X, Y, and T be as above such that the stabilizer of each vertex of X is trivial. Then, G is a quasifree group.*

Proof. By [4, Theorem 3.6], G has the presentation

$$\begin{aligned} \Big\langle G_v, y, x \mid \operatorname{rel} G_v,\ G_m = G_{\bar{m}},\ y \cdot [y]^{-1} G_y [y] \cdot y^{-1} = G_y, \\ x \cdot G_x \cdot x^{-1} = G_x,\ x^2 = [x]^2 \Big\rangle \end{aligned} \tag{3.3}$$

via the map $G_v \to G_v$, $y \to [y]$, and $x \to [x]$ where $v \in V(T)$, $m \in Y_0$, $y \in Y_1$, and $x \in Y_2$.

Since the stabilizer of each vertex of X is trivial, G_v, G_m, G_y, and G_x are trivial for all $v \in V(T)$, $m \in Y_0$, $y \in Y_1$, and $x \in Y_2$. This implies that $[x]^2 = 1$ for all $x \in Y_2$. Then, G has the presentation $\langle y,x \mid x^2 = 1\rangle$, where $y \in Y_1$ and $x \in Y_2$. Then, G is a free product of C_∞ generated by y, and C_2 generated by x.

This implies that G is a quasifree group. This completes the proof. □

Corollary 3.5. *Let G, X, Y, and T be as above, and H a subgroup of G such that $H \cap G_v$ is trivial for all $v \in V(X)$. Then H is a quasifree group.*

Definition 3.6. For each edge y of Y, define the following.

(1) Define $-y$ to be the edge $-y = [y]^{-1}(y)$ if $o(y) \in V(T)$, otherwise $-y = y$.

(2) Define $+y$ to be the edge $+y = [y](-y)$.

It is clear that $t(-y) = (t(y))^*$, $o(+y) = (o(y))^*$, $G_{-y} \le G_{(t(y))^*}$, and $G_{+y} \le G_{(o(y))^*}$. Moreover, if $y \in Y_0 \cup Y_2$, then $G_{-y} = G_{+y} = G_y$.

(3) Define ϕ_y to be the map $\phi_y : G_{-y} \to G_{+y}$ given by $\phi_y(g) = [y]g[y]^{-1}$.

It is clear that ϕ_y is an isomorphism.

(4) Define δ_y to be the element $\delta_y = [y][\bar{y}]$.

It is clear that $\delta_y = 1$ if $y \in Y_0 \cup Y_1$, and $\delta_y = [y]^2$ if $y \in Y_2$. Consequently, $\delta_y \in G_y$, $\delta_{\bar{y}} = \delta_y$, and $\phi_y(\delta_y) = \delta_y$.

Definition 3.7. By a word w of G, we mean an expression of the form $w = g_0 \cdot y_1 \cdot g_1 \cdot y_2 \cdot g_2 \cdots y_n \cdot g_n$, $n \ge 0$, $y_i \in E(Y)$, for $i = 1, 2, \ldots, n$ such that

(1) $g_0 \in G_{(o(y_1))^*}$,

(2) $g_i \in G_{(t(y_i))^*}$ for $i = 1, 2, \ldots, n$,

(3) $(t(y_i))^* = (o(y_{i+1}))^*$ for $i = 1, 2, \ldots, n-1$.

If $w = 1$, then w is called a trivial word of G.

Let w be the word defined above. We have the following concepts:

(a) w is called reduced if w contains no expression of the form $y_i \cdot g_i y_i^{-1}$ if $g_i \in G_{-y_i}$, or $y_i \cdot g_i \cdot y_i$ if $g_i \in G_{y_i}$ and $G(y_i, \bar{y}_i) \ne \varnothing$;

(b) we define $o(w) = (o(y_1))^*$ and $t(w) = (t(y_n))^*$. If $o(w) = t(w) = v$, then w is called a closed word of G of type v;

(c) the value of w denoted by $[w]$ is defined as the element $[w] = g_0[y_1]g_1[y_2]g_2 \cdots [y_n]g_n$ of G;

(d) n is called the length of w and is denoted by $|w| = n$;

(e) the inverse of w denoted by w^{-1} is defined as the word $w^{-1} = g_n^{-1} \cdot \bar{y}_n \cdot \delta_{y_n}^{-1} g_{n-1}^{-1} \cdots g_2^{-1} \cdot \bar{y}_2 \cdot \delta_{y_2}^{-1} g_1^{-1} \cdot \bar{y}_1 \cdot \delta_{y_1}^{-1} g_0^{-1}$ of G.

It is clear that $[w^{-1}] = [w]^{-1}$ but $(w^{-1})^{-1} \ne w$ if w contains an edge y (say) such that $G(y, \bar{y}) \ne \varphi$. Otherwise, $(w^{-1})^{-1} = w$.

Proposition 3.8. *Every element of G is the value of a closed and reduced word of G. Moreover, if w is a nontrivial closed and reduced word of G, then $[w]$ is not the identity element of G. Moreover, if w_1 and w_2 are two closed and reduced words of G such that w_1 and w_2 are of the same type and of the same value, then $|w_1| = |w_2|$.*

Proof. See [5, Corollary 3.6]. □

4. The main result.

The main result of this section is the following theorem.

Theorem 4.1. *Let G be a group acting on a tree X with inversions such that each edge stabilizer is malnormal in G. Let g be a nontrivial element of G and $C(g)$ the centralizer of g in G. Then,*

(i) *if g is in a vertex stabilizer, then $C(g)$ is in that vertex stabilizer;*

(ii) *if g is not in any vertex stabilizer, then $C(g)$ is an infinite cyclic subgroup of G if g does not transfer an edge of X to its inverse. Otherwise, $C(g)$ is a finite cyclic subgroup of G of order 2.*

Proof. (i) Let v be a vertex of X such that $g \in G_v$. We need to show that $C(g)$ is contained in G_v. Let f be an element of G such that $fg = gf$. We need to show that f is in G_v. We consider two cases.

Case 1 (g is in G_x, where x is an edge of X such that v is an end of x). Since the edge stabilizer for each edge of X is malnormal in G, then G_x is malnormal in G_v. This implies that f is in G_x. Consequently, f is in G_v or, equivalently, $C(g)$ is contained in G_v.

Case 2 (g is not in any edge stabilizer of G). Then, there exists a unique vertex v^* of T such that $G(v,v^*) \neq \emptyset$, that is, v and v^* are in the same G vertex orbit. Then, there exist $a \in G$ and $b \in G_{v^*}$ such that $g = aba^{-1}$. This implies that $v = a(v^*)$ and $a^{-1}fab = ba^{-1}fa$. Let $h = a^{-1}fa$. Since $g \notin G_x$ for all $x \in E(X)$, $t(x) = v$, therefore $h \notin G_{-y}$, for all $y \in E(Y)$ such that $(t(y))^* = v$. By Proposition 3.8, there exists a reduced word $w = g_0 \cdot y_1 \cdot g_1 \cdots y_n \cdot g_n$ of G such that w is of type v^* and of value h. That is, $(o(y))^* = (t(y))^* = v^*$ and $[w] = h$. Since $h \notin G_{-y_n}$, then $w \cdot h$ and $h \cdot w$ are reduced words of G of value 1, the identity element of G. Therefore, by Proposition 3.8, the word $w \cdot h \cdot w^{-1} \cdot h^{-1} = g_0 \cdot y_1 \cdot g_1 \cdots y_n \cdot g_n h g_n^{-1} \cdot \bar{y}_n \cdot \delta_{y_n}^{-1} g_{n-1}^{-1} \cdots g_2^{-1} \cdot \bar{y}_2 \cdot \delta_{y_2}^{-1} g_1^{-1} \cdot \bar{y}_1 \cdot \delta_{y_1}^{-1} g_0^{-1} h^{-1}$ is not reduced. The only way that the indicated word can fail to be reduced is that $g_n h g_n^{-1} \in G_{-y_n}$. Replacing a subword of the form $y_i \cdot g_i \cdot \bar{y}_i$ if $g_i \in G_{-y_i}$ by $\phi_{y_i}(g_i \delta_{y_i})$, or replacing a subword of the form $y_i \cdot g_i \cdot y_i$ if $g_i \in G_{y_i}$ and $G(y_i, \bar{y}_i) \neq \emptyset$, by $\phi_{y_i}(g_i \delta_{y_i})$, we see that each $L_i = g_{n-i} \phi_{y_{n-i+1}}(L_{i-1}) g_{n-i}^{-1}$ is in $G_{-y_{n-i}}$ for $i = 1, \ldots, n$ with the convention that $L_0 = g_n h g_n^{-1}$ and $L_n = g_0 \phi_{y_1}(L_{n-1}) g_0^{-1} h^{-1} = 1$. Then $h = g_0 \phi_{y_1}(L_{n-1}) g_0^{-1}$.

Since $(o(y_1))^* = v^*$, $\phi_{y_1}(L_{n-1}) \in G_{+y_1} \leq G_{v^*}$, and $g_0 \in G_{v^*}$, then $h \in G_{v^*}$. This implies that $a^{-1}fa \in G_{v^*}$. Therefore, $f \in aG_{v^*}a^{-1} = G_v$. Consequently $C(g)$ is contained in G_v.

(ii) Now, suppose that g is not in any vertex stabilizer of G. Then, $C(g)$ has trivial intersection with each vertex stabilizer of G. If $a \neq 1$ is in $C(g)$, and a is in a vertex stabilizer G_v of G, for the vertex v of X, then g is in $C(a)$ and, by above, g is in G_v. This contradicts the assumption that g is not in any vertex stabilizer of G. Hence, by Corollary 3.5, $C(g)$ is a free product of a number of infinite cyclic groups and a number of finite cyclic groups of order 2. Since $C(g)$ has a nontrivial center, and the center (see [3, Corollary 4.5, page 211]) of free product of groups of more than one factor is trivial, then $C(g)$ is an infinite cyclic groups, or $C(g)$ is a finite cyclic groups of order 2. If g transfers an edge of X to its inverse, then, by [7, Corollary 4.3], $C(g)$ is a finite cyclic group of order 2. Otherwise, $C(g)$ is an infinite cyclic group. This completes the proof. □

Definition 4.2. Let n be a positive integer and g a nontrivial element of the group H. We say that g has at most nth root if whenever $g = a^n = b^n$, for a, b in H, then $a = b$.

In the next corollaries, the group G satisfies the hypothesis of Theorem 4.1.

Corollary 4.3. *Any element of G that is not in any vertex stabilizer of G has at most nth root.*

Proof. Let g, a, and b be elements of G such that g is not in any vertex stabilizer of G, and $g = a^n = b^n$. We need to show that $a = b$. By Theorem 4.1, $C(g)$ is an infinite cyclic group or is a finite cyclic group of order 2.

Since $ga = ag$ and $gb = bg$, then a and b are in $C(g)$. Then, it is clear that $g = a^n = b^n$ implies that $a = b$. This completes the proof. □

Corollary 4.4. *Let g be an element of G. Then, g is not in any vertex stabilizer of G if and only if g^n is not in any vertex stabilizer of G, where n is a positive integer.*

Proof. Since g^n commutes with g, then g^n is in $C(g)$ which, by Theorem 4.1, is not in any vertex stabilizer of G and the result follows. This completes the proof. □

Corollary 4.5. *Let f and g be two elements of G, and m and n two positive integers such that f and g are not in any vertex stabilizer of G and $f^m g^n = g^n f^m$. Then $fg = gf$.*

Proof. From Corollary 4.4, we get

$$\begin{aligned} f^m g^n = g^n f^m &\Longrightarrow f^m g^n f^{-m} = g^n \\ &\Longrightarrow (f^m g f^{-m})^n = g^n \\ &\Longrightarrow f^m g f^{-m} = g \\ &\Longrightarrow f^m = g f^m g^{-1} \\ &\Longrightarrow f^m = (g f g^{-1})^m \\ &\Longrightarrow f = g f g^{-1} \\ &\Longrightarrow fg = gf. \end{aligned} \tag{4.1}$$

This completes the proof. □

Corollary 4.6. *Let f and g be two elements of G such that f and g are not in any vertex stabilizer of G and $f \in C(g)$. Then $C(f) = C(g)$.*

Proof. By Theorem 4.1, $C(f)$ and $C(g)$ are cyclic subgroups of G. Then, there exist two elements a and b of G such that a and b are not in any vertex stabilizer of G, $C(f) = \langle a\rangle$, and $C(g) = \langle b\rangle$. It is clear that if $a \in C(g)$, then $C(f) = C(g)$. There exist two positive integers m and n such that $f = a^m$ and $g = b^n$. Since $fg = gf$, $a^m b^n = b^n a^m$. Then, Corollary 4.5 implies that $ab = ba$. This implies that $ab^n = b^n a$. Then $a \in C(b^n) = C(g)$. This completes the proof. □

5. Applications. This section is an application of Theorem 4.1 and its corollaries. Free groups, free product of groups, free product of groups with amalgamation subgroup, tree product of groups, and HNN groups are examples of groups acting on trees without inversions. A new class of groups called quasi-HNN groups, defined in [2], are examples of groups acting on trees with inversions. In fact, free product of groups, free product of groups with amalgamation subgroup are special cases of tree product of groups and free groups and HNN groups are special cases of quasi-HNN groups.

Proposition 5.1. *Let $G = \prod_{i\in I}^{*}(A_i, U_{jk} = U_{kj})$ be a nontrivial tree product of the groups A_i, $i \in I$, such that U_{ij} are malnormal subgroups of G. Let g be a nontrivial element of G and $C(g)$ be the centralizer of g in G. Then,*

(i) *$C(g)$ is in a conjugate of A_i for some i, $i \in I$, if g is in a conjugate of A_i;*

(ii) *if g is not in a conjugate of A_i, for all $i \in I$, then $C(g)$ is an infinite cyclic group and g has at most nth root.*

Proof. By [6], there exists a tree X on which G acts without inversions such that any tree of representatives for the action of G on X equals the corresponding fundamental domain for the action of G on X, and for every vertex u of X and every edge x of X, G_u is isomorphic to A_i, $i \in I$, and G_x is isomorphic to U_{ik} for some i, k in I. Moreover, G contains no invertor elements. Therefore, by Theorem 4.1 and Corollary 4.3, the proof of Proposition 5.1 follows. This completes the proof. □

Corollary 5.2. *Let $G = A \underset{C}{*} B$ be a free product of the groups A and B with amalgamation subgroup C such that C is a malnormal subgroup of G. Let g be a nontrivial element of G and $C(g)$ the centralizer of g in G. Then,*

(i) *$C(g)$ is in a conjugate of A or B if g is in a conjugate of A or B;*

(ii) *if g is not in a conjugate of A or B, then $C(g)$ is an infinite cyclic group and g has at most nth root.*

Corollary 5.3. *Let $G = A * B$ be a free product of the groups A and B. Let g be a nontrivial element of G and $C(g)$ the centralizer of g in G. Then,*

(i) *$C(g)$ is in a conjugate of A or B if g is in a conjugate of A or is in a conjugate of B;*

(ii) *if g is not in a conjugate of A or B, then $C(g)$ is an infinite cyclic group and g has at most nth root.*

Proposition 5.4. *Let G^* be the quasi-HNN group*

$$G^* = \langle G, t_i, t_j \mid \operatorname{rel} G,\ t_i A_i t_i^{-1} = B_i,\ t_j C_j t_j^{-1} = C_j,\ t_j^2 = c_j,\ i \in I,\ j \in J \rangle \tag{5.1}$$

such that A_i, B_i, and C_j, $i \in I$ and $j \in J$, are malnormal subgroups of G^*. Let g a nontrivial element of G^* and $C(g)$ the centralizer of g in G^*. Then,

(i) *$C(g)$ is in a conjugate of G if g is in a conjugate of G;*

(ii) *if g is not in a conjugate of G, then $C(g)$ is an infinite cyclic group or a finite cyclic group of order 2 and g has at most nth root.*

Proof. By [8, Lemma 5.1], there exists a tree X on which G^* acts with inversions such that G^* is transitive on $V(X)$, and for every vertex v of X and every edge x of X, G_v^* is isomorphic to G and G_x^* is isomorphic to A_i, $i \in I$, or isomorphic to C_j, $j \in J$. Moreover, G^* contains the invertor elements conjugate to an element t_j, $j \in J$. Therefore, by Theorem 4.1, the proof of Proposition 5.4 follows. This completes the proof. □

Corollary 5.5. *Let G^* be the HNN group*

$$G^* = \langle G, t_i \mid \operatorname{rel} G,\ t_i A_i t_i^{-1} = B_i,\ i \in I \rangle \tag{5.2}$$

such that A_i and B_i, $i \in I$, are malnormal subgroups of G^*. Let g a nontrivial element of G^* and $C(g)$ the centralizer of g in G^*. Then,

(i) *$C(g)$ is in a conjugate of G if g is in a conjugate of G;*

(ii) *if g is not in a conjugate of G, then $C(g)$ is an infinite cyclic group and g has at most nth root.*

Corollary 5.6. *If g is a nontrivial element of a free group F, then the centralizer $C(g)$ of g in F is an infinite cyclic group and g has at most nth root.*

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